Twist Fields in Massive Dirac Theory

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Outline

1. Twist Fields
2. CFT
3. Dirac Theory
4. Correlation Functions
5. Future Work
Use Euclidean time $y = it$ and define complex coordinate

$$z = \frac{-i}{2}(x + iy)$$

Dirac field with mass $m$ satisfies

$$\bar{\partial}\Psi_R = -im\Psi_L, \quad \partial\Psi_L = im\Psi_R$$

as well as the equal time anti-commutation relations

$$\left\{ \Psi_R(x_1), \Psi_R^\dagger(x_2) \right\} = 4\pi\delta(x_1 - x_2) \quad \left\{ \Psi_L(x_1), \Psi_R^\dagger(x_2) \right\} = 0$$
Dirac theory has $U(1)$ symmetry $\Psi \mapsto e^{2\pi i \alpha} \Psi$. [Sato et al., 1979] Associated primary twist field $\sigma_\alpha(x, y)$ has the following properties:

- local
- spin-less
- $U(1)$ neutral
- scaling dimension $\alpha^2$
- $\langle \sigma_\alpha \rangle = c_\alpha m^{\alpha^2}$
**Twist Property**

Inside time ordered correlation functions

$$C(x, y) = \langle \text{vac} | \mathcal{T} [\cdots \Psi_{R,L} \sigma_\alpha(x, y) \cdots ] | \text{vac} \rangle$$

they have the monodromy property

$$C(e^{2\pi i z}) = e^{-2\pi i \alpha} C(z).$$

Which can also be expressed via the equal-time exchange relations

$$\Psi_{R,L}(x) \sigma_\alpha(0) = \begin{cases} \sigma_\alpha(0) \Psi_{R,L}(x) & (x < 0) \\ e^{2\pi i \alpha} \sigma_\alpha(0) \Psi_{R,L}(x) & (x > 0) \end{cases}$$

$$\Psi^\dagger_{R,L}(x) \sigma_\alpha(0) = \begin{cases} \sigma_\alpha(0) \Psi^\dagger_{R,L}(x) & (x < 0) \\ e^{-2\pi i \alpha} \sigma_\alpha(0) \Psi^\dagger_{R,L}(x) & (x > 0). \end{cases}$$

This gives the form-factors of the twist fields [Karowski and Weisz, 1978].
Massless Limit of Dirac Theory

$\Psi_{R(L)}$ becomes (anti-)holomorphic and two point functions are

$$\left\{ \Psi_R^\dagger(x_1, x_2), \Psi_R(x_2, y_2) \right\} = \frac{1}{z_1 - z_2}.$$

We reproduce these correlation functions via bosonisation.

**Definition ($e_\alpha(z)$)**

These are bosonic operators with index $\alpha \in \mathbb{R}$ and correlators

$$\langle \text{vac}| e_{\alpha_1}(x_1, y_1) \cdots e_{\alpha_n}(x_n, y_n)|\text{vac}\rangle_{m=0} = \delta_{0, \sum_j \alpha_j} \prod_{j<k} (-i(z_j - z_k))^{\alpha_j \alpha_k}.$$
Properties of $e_\alpha$

These correlation functions imply the OPE:

$$\mathcal{T} \left[ e_\alpha(x, y)e_{\alpha'}(x', y') \right] \sim (-i(z-z'))^{\alpha\alpha'} \left( 1 + \frac{\alpha}{\alpha + \alpha'}(z - z') \partial' \right) e_{\alpha + \alpha'}(x', y')$$

and the equal-time exchange relations

$$e_{\alpha_1}(x_1)e_{\alpha_2}(x_2) = e^{i\pi \alpha_1 \alpha_2 \text{sgn}(x_2-x_1)} e_{\alpha_2}(x_2)e_{\alpha_1}(x_1).$$

So we can identify

$$\Psi_R = -e^{-i\pi/4} \omega^{-1} \hat{j} e_{-1}, \quad \Psi_L = e^{i\pi/4} \omega \hat{i} \bar{e}_1.$$
Reproducing Twist Fields

Setting the $U(1)$ charge to be $Q$ such that

$$\left[ Q, \Psi_{R,L}^\dagger \right] = -\Psi_{R,L}^\dagger \quad \quad \left[ Q, \Psi_{R,L} \right] = \Psi_{R,L}$$

and its action on the $e_\alpha$ and $\bar{e}_\alpha$ is given by

$$e^{i\alpha Q} e_\alpha e^{-i\alpha Q} = e^{-i\alpha} e_\alpha \quad \quad e^{i\alpha Q} \bar{e}_\alpha e^{-i\alpha Q} = e^{i\alpha} \bar{e}_\alpha,$$

the twist field can be identified as

$$\sigma_\alpha = e_\alpha \bar{e}_\alpha e^{-i\pi \alpha Q}$$
New fields

We can now use the OPE for $e_\alpha$ and $\bar{e}_\alpha$ to identify the descendent fields:

$$\mathcal{T} \left[ \Psi_R^\dagger(x, y) \sigma_\alpha(0) \right] \sim (-iz)^\alpha e^{-i\pi/4} \omega \hat{\jmath} e_{\alpha+1} \bar{e}_\alpha e^{-i\pi\alpha Q}$$

$$\mathcal{T} \left[ \Psi_R(x, y) \sigma_\alpha(0) \right] \sim -(iz)^{-\alpha} e^{-i\pi/4} \omega^{-1} \hat{\jmath} e_{\alpha-1} \bar{e}_\alpha e^{-i\pi\alpha Q}$$

and so

**Definition (Descendent Twist Fields)**

$$\sigma_{\alpha+1,\alpha} := e^{-i\pi/4} \omega \hat{\jmath} e_{\alpha+1} \bar{e}_\alpha e^{-i\pi\alpha Q}$$

$$\sigma_{\alpha-1,\alpha} := -e^{-i\pi/4} \omega^{-1} \hat{\jmath} e_{\alpha-1} \bar{e}_\alpha e^{-i\pi\alpha Q}$$

Note that the same two families of fields arise consistently from the OPEs of $\Psi_L^{(\dagger)}$ with $Q$. 
Dirac Theory
Form Factors of Descendent Fields

Calculate these via e.g.

$$\langle \text{vac} | \Psi_R^\dagger (x, y) \sigma_\alpha (0) | \vartheta \rangle_+$$

and looking for the part which diverges like $z^\alpha$.

Insert identity

$$1 = \sum_{N=0}^\infty \frac{1}{N!} \sum_{\varepsilon_1 \cdots \varepsilon_N} \int_{-\infty}^\infty d\vartheta_1 \cdots \int_{-\infty}^\infty d\vartheta_N \langle \vartheta_1 \cdots \vartheta_N | \varepsilon_1 \cdots \varepsilon_N \rangle_{\varepsilon_1 \cdots \varepsilon_N}^{\varepsilon_N \cdots \varepsilon_1} \langle \vartheta_1 \cdots \vartheta_N | \vartheta_1 \rangle.$$

between fields and only the one particle term survives.
One Particle Form Factor

We set $y = 0$ for convenience.

$$\langle \text{vac} | \Psi_R^\dagger(x) | \vartheta \rangle_+ = \sqrt{m} e^{\vartheta/2} e^{i x \vartheta}$$

$$+ \langle \varphi | \sigma_\alpha(0) | \vartheta \rangle_+ = c_\alpha m^{\alpha^2} \frac{\sin(\pi \alpha)}{2\pi} e^{-i\pi \alpha} e^{\alpha(\vartheta - \varphi)} \frac{e^{\alpha(\vartheta - \varphi)}}{\sinh \left( \frac{\vartheta - \varphi + i0^+}{2} \right)}$$

$$+ c_\alpha m^{\alpha^2} \delta(\vartheta - \varphi)$$

From these the term which contains all the divergence can be written

$$c_\alpha m^{\alpha^2 + 1/2} \frac{\sin(\pi \alpha)}{\pi} e^{-i\pi \alpha/2} \int d\varphi \frac{e^{\alpha \vartheta - (\alpha - 1/2)\varphi + x E_{\varphi}}}{i e^{\frac{\vartheta - \varphi}{2}} - e^{\frac{\varphi - \vartheta}{2}}}$$
\begin{align*}
\langle \text{vac} | \sigma_{\alpha+1, \alpha}(0) | \vartheta \rangle_+ &= c_\alpha \frac{e^{-i\pi\alpha/2}}{\Gamma(1 + \alpha)} m^{\alpha^2 + \alpha + 1/2} e^{(\alpha+1/2)\vartheta} \\
\text{This follows from our definitions but following the same method we may also calculate} \nonumber \\
\langle \text{vac} | \sigma_{\alpha, \alpha-1}(0) | \vartheta \rangle_+ &= ic_\alpha \frac{e^{-i\pi\alpha/2}}{\Gamma(1 - \alpha)} m^{\alpha^2 - \alpha + 1/2} e^{\vartheta(\alpha-1/2)}
\end{align*}

**Recursion Relation for** $c_\alpha$

\[
\frac{c_\alpha}{c_{\alpha+1}} = \frac{\Gamma(\alpha + 1)}{\Gamma(-\alpha)}
\]

which agrees with previous results [Lukyanov and Zamolodchikov, 1997, Doyon, 2003] giving

\[
c_\alpha = \frac{1}{G(1 - \alpha)G'(1 + \alpha)}
\]
Another Consistency Check

From our CFT arguments we see that

\[ \sigma_{\alpha+1}(0) \sim i(-iz)^{-\alpha} \mathcal{T}[\Psi_R^\dagger(x, y)\sigma_{\alpha,\alpha+1}(0)] \]

So following the same method we find

\[ \langle \text{vac} | \Psi_R^\dagger(x, 0)\sigma_{\alpha,\alpha+1}(0) | \text{vac} \rangle \sim -ic_\alpha m^{\alpha^2 + \alpha + 1} \frac{\Gamma(-\alpha)}{\Gamma(1 + \alpha)} \left( \frac{2}{-mx} \right)^{-\alpha} \]

and as

\[ \langle \sigma_{\alpha+1} \rangle = c_{\alpha+1} m^{(\alpha+1)^2} \]

we find the same recursion relation and have another consistency check on our new fields.
Double Model

To evaluate correlation functions we consider two non-interacting copies of the Dirac Theory $\Psi$ and $\Phi$ in analogy to the Ising case [Fonseca and Zamolodchikov, 2003]. We are interested in 3 specific conserved charges: $P$ and $\bar{P}$ related to conserved momentum and $Z$ related to the $O(2)$ rotation symmetry amongst the copies.

**The action of $P$ and $\bar{P}$**

\[
[P, O^\Psi O^\Phi] = i\partial O^\Psi O^\Phi - iO^\Psi \partial O^\Phi
\]

\[
[\bar{P}, O^\Psi O^\Phi] = -i\bar{\partial} O^\Psi O^\Phi + iO^\Psi \bar{\partial} O^\Phi
\]

**The action of $Z$**

\[
[Z, \Psi] = -\Phi \\
[Z, \Phi] = \Psi
\]
We can construct the double model $Z$ in the CFT and find, for example,

\[
\begin{align*}
[Z, \sigma_\alpha^\Psi \sigma_\alpha^\Phi] &= 0 \\
[Z, \sigma_\alpha^\Psi \sigma_{\alpha-1}^\Phi] &= i(\sigma_{\alpha-1,\alpha}^\Psi \sigma_{\alpha,\alpha-1}^\Phi - \sigma_{\alpha,\alpha-1}^\Psi \sigma_{\alpha-1,\alpha}^\Phi) \\
[Z, \sigma_\alpha^\Psi \sigma_{\alpha+1}^\Phi] &= i(\sigma_{\alpha+1,\alpha}^\Psi \sigma_{\alpha,\alpha+1}^\Phi - \sigma_{\alpha,\alpha+1}^\Psi \sigma_{\alpha+1,\alpha}^\Phi)
\end{align*}
\]

These relations have been verified using the form factors expansions up to 4 particles.
Ward Identities

Given translation and parity invariance the Ward identities below reduce to a set of PDEs for the correlation functions $\langle \sigma_\alpha(x, y) \sigma_\alpha(0) \rangle$, $\langle \sigma_\alpha(x, y) \sigma_{\alpha+1}(0) \rangle$ and $\langle \sigma_{\alpha+1,\alpha}(x, y) \sigma_{\alpha,\alpha+1}(0) \rangle$

\[
\langle [Z, \sigma_\alpha^\Psi (x, y) \sigma_{\alpha+1}^\Phi (x, y)] \sigma_{\alpha+1,\alpha}^\Psi (0) \sigma_{\alpha,\alpha+1}^\Phi (0) \rangle = 0
\]

\[
\langle [Z, [P, \sigma_\alpha^\Psi (x, y) \sigma_{\alpha+1}^\Phi (x, y)]] \sigma_{\alpha+1,\alpha}^\Psi (0) \sigma_{\alpha,\alpha+1}^\Phi (0) \rangle = 0
\]

\[
\langle [Z, [\bar{P}, \sigma_\alpha^\Psi (x, y) \sigma_{\alpha+1}^\Phi (x, y)]] \sigma_{\alpha,\alpha+1}^\Psi (0) \sigma_{\alpha+1,\alpha}^\Phi (0) \rangle = 0
\]

\[
\langle [Z, [P, [\bar{P}, \sigma_\alpha^\Psi (x, y) \sigma_{\alpha+1}^\Phi (x, y)]]] \sigma_{\alpha,\alpha+1}^\Psi (0) \sigma_{\alpha+1,\alpha}^\Phi (0) \rangle = 0
\]

\[
\langle [Z, [P, \sigma_\alpha^\Psi (x, y) \sigma_{\alpha+1}^\Phi (x, y)] [\bar{P}, \sigma_{\alpha+1,\alpha}^\Psi (0) \sigma_{\alpha,\alpha+1}^\Phi (0)]] \rangle = 0
\]

An algebraic relation

\[
e^{2\pi i \alpha} \langle \sigma_{\alpha+1,\alpha}(x, y) \sigma_{\alpha,\alpha+1}(0) \rangle^2 + \langle \sigma_{\alpha+1}(x, y) \sigma_{\alpha}(0) \rangle^2 - \langle \sigma_{\alpha}(x, y) \sigma_{\alpha}(0) \rangle^2 = 0
\]
It is possible to manipulate the previous equations to obtain our main result. Setting \( \langle \sigma_{\alpha}(x, y) \sigma_{\alpha}(0) \rangle = c_{\alpha}^2 m^{2\alpha^2} e^{\Sigma(x, y)} \) we find

\[
\partial \bar{\partial} \Sigma_{\alpha} = \frac{m^2}{2} (1 - \cosh(2\psi))
\]

\[
\partial \bar{\partial} \psi = \frac{m^2}{2} \sinh(2\psi)
\]

In agreement with [Bernard and LeClair, 1994]
And also setting \( \langle \sigma_{\alpha}(x, y) \sigma_{\alpha+1}(0) \rangle = c_{\alpha} c_{\alpha+1} m^{2\alpha^2 + 2\alpha + 1} e^{\Sigma'(x, y)} \)

\[
\partial \bar{\partial} \Sigma'_{\alpha} = \frac{2 \tanh^2(\psi)}{\cosh(2\psi) - 1} \partial \psi \bar{\partial} \psi.
\]
Future Work
What am I going to do next?

- Differential equations of $\langle \sigma_\alpha(x, y)\sigma_\beta(0) \rangle$
- Fix initial conditions for differential equations
- Applications to twist fields in models with higher symmetry
- Finite Temperature without parity symmetry
Differential equations for sine-Gordon correlation functions at the free fermion point. 

Two-point correlation functions of scaling fields in the Dirac theory on the Poincare disk. 

Ward identities and integrable differential equations in the Ising field theory.

Exact Form-Factors in (1+1)-Dimensional Field Theoretic Models with Soliton Behavior. 
*Nucl. Phys.*, B139:455.

Exact expectation values of local fields in quantum sine- Gordon model. 

HOLONOMIC QUANTUM FIELDS. 4. 