Beta stochastic volatility model

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Introduction. Quoting Jesper Andreasen from ”Risk 25: No more heroes in quantitative finance?”, Risk Magazine, August 2012: ”Quants have hundreds of models and, even in one given asset class, a quant will have 10 models that can fit the smile. The question is, which is the right delta? That’s still an open question, even restricting it to vanilla business. It’s one reason why there’s so little activity in the interest rate options markets.”

Quoting Jim Gatheral from the same article: ”With less trading in exotics and vanillas moving to exchanges, we need to focus on generating realistic price dynamics for underlyings.”

I present a stochastic volatility model that:
1) can be made consistent with different volatility regimes (thus, potentially computing a correct delta)
2) is consistent with observed dynamics of the spot and its volatility
3) has very intuitive model parameters

The model is based on joint article with Piotr Karasinski:
Plan of the presentation

1) Discuss existing volatility models and their limitations

2) Discuss volatility regimes observed in the market

3) Introduce beta stochastic volatility (SV) model

4) Emphasize intuitive and robust calibration of the beta SV model

5) Case study I: application of the beta SV model to model the correlation skew

6) Case study II: application of the beta SV model to model the conditional forward skew
Motivation I. Applications of volatility models

1) Interpolators for implied volatility surface
- Represent functional forms for implied volatility at different strikes and maturities
- Assume no dynamics for the underlying (”apart from the SABR model”)
- Applied for marking vanilla options and serve as inputs for calibration of dynamic volatility models (local vol, stochastic vol)

2) Hedge computation for vanilla options
- Apply deterministic rules for changes in model parameters given change in the spot
- The most important is the volatility backbone - the change in the ATM volatility (and its term structure) given change in the spot price
- Applied for computation of hedges for vanilla and exotic books

3) Dynamics models (local vol, stochastic vol, local stochastic vol)
- Compute the present value and hedges of exotic options given inputs from 1) and 2)
Motivation II. Missing points

1) Interpolators
- Do not assume any specific dynamics
- Provide a tool to compute the market observables from given snapshot of market data: at-the-money (ATM) volatility, skew, convexity, and term structures of these quantities

2) Vanilla hedge computation
- Assume specific functional rules for changes in market observables given changes in market data

3) Dynamic models (local vol, Heston) for pricing exotic options
- No explicit connection to market observables and their dynamics
- Apply "blind" non-linear and non-intuitive fitting methods for calibration of model parameters (correlation, vol-of-vol, etc)

We need a dynamic volatility model that could connect all three tools in a robust and intuitive way!
Motivation III. Beta stochastic volatility model
Propose the beta stochastic volatility model that:

1) In its simplified form, the model can be used as an interpolator
   Takes market observables (ATM volatility and skew) for model calibration with model parameters easily interpreted in terms of market observable - volatility skew (with a good approximation)

2) The model is consistent with vanilla hedge computations - it has a model parameter to replicate the volatility backbone (with a good approximation)
   The model assumes the dynamics of the ATM volatility specified by the volatility backbone

3) The model provides robust dynamics for exotic options:
   It produces steep forward skews, mean-reversion
   The model has a mean-reversion and volatility of volatility
Implied volatility skew I
First I describe implied volatility skew and volatility regimes

For time to maturity $T$, the implied volatility, which is applied to value vanilla options using the Black-Scholes-Merton (BSM) formula, can be parameterized by a linear function $\sigma(K; S_0)$ of strike price $K$:

$$\sigma(K; S_0) = \sigma_0 + \beta \left( \frac{K}{S_0} - 1 \right)$$

$\beta, \beta < 0$, is the slope of the volatility skew near the ATM strike

Let's take strikes at $K_{\pm} = (1 \pm \alpha)S_0$, where typically $\alpha = 5\%$:

$$\beta = \frac{1}{2\alpha} \left( \sigma((1 + \alpha)S_0; S_0) - \sigma((1 - \alpha)S_0; S_0) \right) \equiv \text{Skew}_\alpha$$

where $\text{Skew}_\alpha$ is the implied skew normalized by strike width $\alpha$

The equity volatility skew is negative, as consequence of the fact that, relatively, it is more expensive to buy an OTM put option than an OTM call option.
**Implied volatility skew II** for 1m options on the S&P 500 index from October 2007 to July 2012

Left (red): 1m 105% − 95% skew, Skew\(_{5\%}(t_n)\)
Right (green): 1m ATM implied volatility

\(\beta = -1.0\) means that the implied volatility of the put struck at 95% of the spot price is \(-5\% \times \beta = 5\%\) higher than that of the ATM option.
Sticky rules (Derman) I

1) Sticky-strike:

\[ \sigma(K; S) = \sigma_0 + \beta \left( \frac{K}{S_0} - 1 \right) \], \ \sigma_{ATM}(S) \equiv \sigma(S; S) = \sigma_0 + \beta \left( \frac{S}{S_0} - 1 \right) \]

ATM vol increase as the spot declines - typical of range-bounded markets

2) Sticky-delta:

\[ \sigma(K; S) = \sigma_0 + \beta \left( \frac{K - S}{S_0} \right) \], \ \sigma_{ATM}(S) = \sigma_0 \]

The level of the ATM volatility does not depend on spot price - typical of stable trending markets

3) Sticky local volatility:

\[ \sigma(K; S) = \sigma_0 + \beta \left( \frac{K + S}{S_0} - 2 \right) \], \ \sigma_{ATM}(S) = \sigma_0 + 2\beta \left( \frac{S}{S_0} - 1 \right) \]

ATM vol increase as the spot declines twice as much as in the sticky strike case - typical of stressed markets
Sticky rules II

Given: \( \beta = -1.0 \) and \( \sigma_{ATM}(0) = 25.00\% \)

Spot change: down by \(-5\%\) from \(S(0) = 1.00\) to \(S(1) = 0.95\)

Sticky-strike regime: the ATM volatility moves along the original skew increasing by \(-5\% \times \beta = 5\%\)

Sticky-local regime: the ATM volatility increases by \(-5\% \times 2\beta = 10\%\) and the volatility skew moves upwards

Sticky-delta regime: the ATM volatility remains unchanged with the volatility skew moving downwards
Impact on option delta

The key implication of the volatility rules is the impact on option delta $\Delta$

We can show the following rule for call options:

$$\Delta_{Sticky-Local} \leq \Delta_{Sticky-Strike} \leq \Delta_{Sticky-Delta}$$

As a result, for hedging call options, one should be over-hedged (as compared to the BSM delta) in a trending market and under-hedged in a stressed market

Thus, the identification of market regimes plays an important role to compute option hedges

While computation of hedges is relatively easy for vanilla options and can be implemented using the BSM model, for path-dependent exotic options, we need a dynamic model consistent with different volatility regimes
**Stickiness ratio I**

To identify volatility regimes we introduce the stickiness ratio

Given price return from time $t_{n-1}$ to $t_n$:

$$X(t_n) = \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}$$

We make prediction for change in the ATM volatility:

$$\sigma_{ATM}(t_n) = \sigma_{ATM}(t_{n-1}) + \beta R(t_n) X(t_n)$$

where the stickiness ratio $R(t_n)$ indicates the rate of change in the ATM volatility predicted by the skew and price return

Stickiness ratio $R$ is a model-dependent quantity, informally:

$$R \approx \frac{1}{\beta} \frac{\partial}{\partial S} \sigma_{ATM}(S)$$

We obtain that:

$R = 1$ under sticky-strike

$R = 0$ under sticky-delta

$R = 2$ under sticky-local vol
**Stickiness ratio II**

Empirical test is based on using market data for S&P500 (SPX) options from 9-Oct-07 to 1-Jul-12 divided into three zones

<table>
<thead>
<tr>
<th></th>
<th>crisis</th>
<th>recovery</th>
<th>range-bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>start date</td>
<td>9-Oct-07</td>
<td>5-Mar-09</td>
<td>18-Feb-11</td>
</tr>
<tr>
<td>end date</td>
<td>5-Mar-09</td>
<td>18-Feb-11</td>
<td>31-Jul-12</td>
</tr>
<tr>
<td>number days</td>
<td>354</td>
<td>501</td>
<td>365</td>
</tr>
<tr>
<td>start SPX</td>
<td>1565.15</td>
<td>682.55</td>
<td>1343.01</td>
</tr>
<tr>
<td>end SPX</td>
<td>682.55</td>
<td>1343.01</td>
<td>1384.06</td>
</tr>
<tr>
<td>return</td>
<td>-56.39%</td>
<td>96.76%</td>
<td>3.06%</td>
</tr>
<tr>
<td>start ATM 1m</td>
<td>14.65%</td>
<td>45.28%</td>
<td>12.81%</td>
</tr>
<tr>
<td>end ATM 1m</td>
<td>45.28%</td>
<td>12.81%</td>
<td>15.90%</td>
</tr>
<tr>
<td>vol change</td>
<td>30.63%</td>
<td>-32.47%</td>
<td>3.09%</td>
</tr>
<tr>
<td>start Skew 1m</td>
<td>-72.20%</td>
<td>-61.30%</td>
<td>-69.50%</td>
</tr>
<tr>
<td>end Skew 1m</td>
<td>-57.80%</td>
<td>-69.50%</td>
<td>-55.50%</td>
</tr>
<tr>
<td>skew change</td>
<td>14.40%</td>
<td>-8.20%</td>
<td>14.00%</td>
</tr>
</tbody>
</table>
Stickiness ratio III

![Graph showing S&P500 and 1m ATM vol over time, with labels for crisis, recovery, range-bound periods.](image-url)
**Stickiness ratio IV**
To test Stickiness empirically, we apply the regression model for parameter $\bar{R}$ within each zone using daily changes:

$$\sigma_{ATM}(t_n) - \sigma_{ATM}(t_{n-1}) = \bar{R} \times \text{Skew}_{5\%}(t_{n-1}) X(t_n) + \epsilon_n$$

where $X(t_n)$ is realized return for day $n$

$\sigma_{ATM}(t_n)$ and $\text{Skew}_{\alpha}(t_n)$ are the ATM volatility and skew observed at the end of the $n$-th day

$\epsilon_n$ is iid normal residuals

Informal definition of the stickiness ratio:

$$R(t_n) = \frac{\sigma_{ATM}(t_n) - \sigma_{ATM}(t_{n-1})}{X(t_n)\text{Skew}_{5\%}(t_{n-1})}$$

We expect that the average value of $R$, $\bar{R}$, as follows:

$\bar{R} = 1$ under the sticky-strike regime
$\bar{R} = 0$ under the sticky-delta regime
$\bar{R} = 2$ under the sticky-local regime
Stickiness ratio (crisis) for 1m and 1y ATM vols

Stickiness for 1m ATM vol, crisis period Oct 07 - Mar 09

Daily change in 1m ATM vol

$y = 1.6327x$

$R^2 = 0.7738$

5%

10%

15%

-15%

1m Skew * price return

Stickiness for 1y ATM vol, crisis period Oct 07 - Mar 09

Daily change in 1y ATM vol

$y = 1.5978x$

$R^2 = 0.8158$

5%

10%

15%

-5%

1y Skew * price return
Stickiness ratio (recovery) for 1m and 1y ATM vols

Stickiness for 1m ATM vol, recovery period Mar 09-Feb 11

\[ y = 1.4561x \]
\[ R^2 = 0.6472 \]

Stickiness for 1y ATM vol, recovery period Mar 09-Feb 11

\[ y = 1.5622x \]
\[ R^2 = 0.6783 \]
Stickiness ratio (range) for 1m and 1y ATM vols

\[ y = 1.3036x \]

\[ R^2 = 0.6769 \]

Daily change in 1m ATM vol

Stickness for 1m ATM vol, range-bnd period Feb11-Aug12

\[ y = 1.4139x \]

\[ R^2 = 0.7228 \]

Daily change in 1y ATM vol

Stickness for 1y ATM vol, range-bnd period Feb11-Aug12

1m Skew * price return

1y Skew * price return
**Stickiness ratio V. Conclusions**

Summary of the regression model:

<table>
<thead>
<tr>
<th></th>
<th>crisis</th>
<th>recovery</th>
<th>range-bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stickiness, 1m</td>
<td>1.63</td>
<td>1.46</td>
<td>1.30</td>
</tr>
<tr>
<td>Stickiness, 1y</td>
<td>1.60</td>
<td>1.56</td>
<td>1.41</td>
</tr>
<tr>
<td>$R^2$, 1m</td>
<td>77%</td>
<td>65%</td>
<td>68%</td>
</tr>
<tr>
<td>$R^2$, 1y</td>
<td>82%</td>
<td>68%</td>
<td>72%</td>
</tr>
</tbody>
</table>

1) The concept of the stickiness is **statistically significant** explaining about 80% of the variation in ATM volatility during crisis period and about 70% of the variation during recovery and range-bound periods.

2) Stickiness ratio is
   - stronger during crisis period, $\bar{R} \approx 1.6$ (closer to sticky local vol)
   - less strong during recovery period, $\bar{R} \approx 1.5$
   - weaker during range-bound period, $\bar{R} \approx 1.35$ (closer to sticky-strike)

3) The volatility regime is typically neither sticky-local nor sticky-strike but rather a combination of both.
Stickiness ratio VI. Time series

1.75  2.00  2.25 Stickiness  
1m Stickeiness, 60d average  
1.25  1.50  
Oct -07  
Dec -07  
Mar -08  
Jun -08  
Sep -08  
Dec -08  
Mar -09  
Jun -09  
Aug -09  
Nov -09  
Feb -10  
May -10  
Aug -10  
Nov -10  
Feb -11  
Apr -11  
Jul -11  
Oct -11  
Jan -12  
Apr -12  
Jul-12

1m Stickeness, 60d average

Stickiness

Oct-07 Dec-07 Mar-08 Jun-08 Sep-08 Dec-08 Mar-09 Jun-09 Aug-09 Nov-09 Feb-10 May-10 Aug-10 Nov-10 Feb-11 Apr-11 Jul-11 Oct-11 Jan-12 Apr-12 Jul-12
**Stickiness ratio VII. Dynamic models A**

Now we consider how to model the stickiness ratio within the dynamic SV models.

The primary driver is change in the spot price, $\Delta S/S$.

The key in this analysis is what happens to the level of model volatility given change in the spot price (for a very nice discussion see ”A Note on Hedging with Local and Stochastic Volatility Models” by Mercurio-Morini, on ssrn.com).

**The model-consistent hedge:**
The level of volatility changes by (approximately): $\text{Skew} \times \Delta S/S$.

**The model-inconsistent hedge:**
The level of volatility remains unchanged.

Implication for the stickiness under pure SV models:
\[
\overline{R} = 2 \quad \text{under the model-consistent hedge}
\]
\[
\overline{R} = 0 \quad \text{under the model-inconsistent hedge}
\]
**Stickiness ratio VII. Dynamic models B**

How to make $\bar{R} = 1.5$ using SV models?
Under the model-consistent hedge: impossible?
Under the model-inconsistent hedge: mix SV with local volatility

**Remedy:** add jump process

Under any spot-homogeneous jump model, $\bar{R} = 0$

The only way to have a model-consistent hedging that fits the desired stickiness ratio is to **mix stochastic volatility with jumps**:
the higher is the stickiness ratio, the lower is the jump premium
the lower is the stickiness ratio, the higher is the jump premium

Jump premium is lower during crisis periods (after a big crash or excessive market panic, the probability of a second one is lower because of realized de-leveraging and de-risking of investment portfolios, central banks interventions)

Jump premium is higher during recovery and range-bound periods (renewed fear of tail events, increased leverage and risk-taking given small levels of realized volatility and related hedging)
Stickiness ratio VII. Dynamic models C

The above consideration explain that the stickiness ratio is stronger during crisis period, $\bar{R} \approx 1.6$ (closer to sticky local vol) weaker during range-bound and recovery periods, $\bar{R} \approx 1.35$ (closer to sticky-strike)

To model this feature within an SV model, we need to specify a proportion of the skew attributed to jumps (see my 2011 presentation for Risk Quant congress and 2012 presentation for Global derivatives)

During crisis periods, the weight of jumps is about 20%
During range-bound and recovery periods, the weight of jumps is about 40%
Beta stochastic volatility I

First, I present a simplified version of the beta stochastic volatility model introduced in Karasinski and Sepp (2012) with no mean-reversion and volatility-of-volatility:

\[
\frac{dS(t)}{S(t)} = \sigma(t)(S(t))^{\beta_S} dW(t), \quad S(0) = S_0
\]

\[
d\sigma(t) = \beta_V \frac{dS(t)}{S(t)}, \quad \sigma(0) = \sigma_0
\]

(1)

where \( S(t) \) is the spot price
\( \sigma(t) \) is instantaneous volatility and \( \sigma_0 \) is initial level of ATM volatility
\( W(t) \) is a Brownian motion - the only source of randomness

To produce the volatility skew and the dependence between the price and implied volatility, the model relies on the two parameters:
\( \beta_S \) is the backbone beta
\( \beta_V \) is the volatility beta

Estimates of \( \beta_S \) and \( \beta_V \) are easily inferred from implied/historical data
Volatility beta

We replicate $\sigma(t)$ by short-term ATM volatility, $\sigma(t) = \sigma_{ATM}(S(t))$ to estimate model parameters by the regression model

Volatility beta $\beta_V$ is a measure of linear dependence between daily returns and changes in the ATM volatility:

$$\sigma_{ATM}(S(t_n)) - \sigma_{ATM}(S(t_{n-1})) = \beta_V \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}$$

Next we examine this regression model empirically
Volatility beta (crisis) for 1m and 1y ATM vols

Daily change in 1m ATM vol, crisis period Oct 07 - Mar 09

Daily change in 1y ATM vol, crisis period Oct 07 - Mar 09

Daily price return
Volatility beta (recovery) for 1m and 1y ATM vols

Daily change in 1m ATM vol, recovery period Mar 09-Feb 11

Daily change in 1y ATM vol, recovery period Mar 09-Feb 11

\[ y = -0.9109x \]
\[ R^2 = 0.603 \]

\[ y = -0.3926x \]
\[ R^2 = 0.6475 \]
Volatility beta (range) for 1m and 1y ATM vols
Volatility beta. Summary

<table>
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<tr>
<th></th>
<th>crisis</th>
<th>recovery</th>
<th>range-bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility beta 1m</td>
<td>-1.11</td>
<td>-0.91</td>
<td>-1.07</td>
</tr>
<tr>
<td>Volatility beta 1y</td>
<td>-0.39</td>
<td>-0.39</td>
<td>-0.44</td>
</tr>
<tr>
<td>$R^2$ 1m</td>
<td>76%</td>
<td>60%</td>
<td>68%</td>
</tr>
<tr>
<td>$R^2$ 1y</td>
<td>80%</td>
<td>65%</td>
<td>71%</td>
</tr>
</tbody>
</table>

The volatility beta is pretty stable across different market regimes.

The longer term ATM volatility is less sensitive to changes in the spot.

Changes in the spot price explain about:
80% in changes in the ATM volatility during crisis period
60% in changes in the ATM volatility during recovery period (ATM volatility reacts slower to increases in the spot price)
70% in changes in the ATM volatility during range-bound period (jump premium start to play bigger role in recovery and range-bound periods)
The backbone beta

The backbone beta $\beta_S$ is a measure of daily changes in the logarithm of the ATM volatility to daily returns on the stock

$$\ln [\sigma_{ATM}(S(t_n))] - \ln [\sigma_{ATM}(S(t_{n-1})] = \beta_S \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}$$

Next we examine this regression model empirically
Backbone beta (crisis) for 1m and 1y ATM vols

Daily change in log 1m ATM vol, crisis period Oct07 - Mar09

\[ y = -2.8134x \]
\[ R^2 = 0.6722 \]

Daily change in log 1m ATM vol, crisis period Oct 07 - Mar 09

\[ y = -1.2237x \]
\[ R^2 = 0.7715 \]
Backbone beta (recovery) for 1m and 1y ATM vols
Backbone beta (range) for 1m and 1y ATM vols

Daily change in log 1m ATM vol, range period Feb11-Aug12

\[ y = -4.5411x \]
\[ R^2 = 0.6212 \]

Daily change in log 1y ATM vol, range period Feb11-Aug12

\[ y = -1.8097x \]
\[ R^2 = 0.6978 \]
The backbone beta. Summary A

<table>
<thead>
<tr>
<th></th>
<th>crisis</th>
<th>recovery</th>
<th>range-bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backbone beta 1m</td>
<td>-2.81</td>
<td>-3.64</td>
<td>-4.54</td>
</tr>
<tr>
<td>Backbone beta 1y</td>
<td>-1.22</td>
<td>-1.43</td>
<td>-1.81</td>
</tr>
<tr>
<td>$R^2$ 1m</td>
<td>67%</td>
<td>54%</td>
<td>62%</td>
</tr>
<tr>
<td>$R^2$ 1y</td>
<td>77%</td>
<td>61%</td>
<td>70%</td>
</tr>
</tbody>
</table>

The value of the backbone beta appears to be less stable across different market regimes (compared to volatility beta)

Explanatory power is somewhat less (by 5-7%) for 1m ATM vols (compared to volatility beta)

Similar explanatory power for 1y volatilities
The backbone beta. Summary B
Change in the level of the ATM volatility implied by backbone beta $\beta_S$ is proportional to initial value of the ATM volatility
High negative value of $\beta_S$ implies a big spike in volatility given a modest drop in the price - a feature of sticky local volatility model

In the figure, using estimated parameters $\beta_V = -1.07$, $\beta_S = 4.54$ in range-bound period, $\sigma(0) = 20\%$
Connection to the SABR model (Hagan et al (2002) )

Model parameters are related to the SABR model as follows:

\[ \tilde{a} = \sigma_0, \quad \rho = -1, \quad \nu = -\beta V, \quad \beta = \beta_S + 1 \]

Using formula (3.1a) in Hagan et al (2002) for a short maturity and small log-moneyness \( k, \ k = \ln(K/S_0) \), we obtain the following relationship for the BSM implied volatility \( \sigma_{\text{IMP}}(k) \):

\[
\sigma_{\text{IMP}}(k) = \frac{\sigma_0}{S^\beta_S} \left[ 1 + \frac{1}{2} \left( \beta_S + \frac{\beta V}{\sigma_0} \right) k + \frac{1}{12} \left( \frac{\beta}{S} - \left( \frac{\beta V}{\sigma_0} \right)^2 \right) k^2 \right]
\]

Thus, in a simple case, the model can be directly linked to the implied volatility interpolator represented by the SABR model.
Model implied skew
We obtain the following approximate but accurate relationship between the model parameters and short-term implied ATM volatility, $\sigma_{ATM}(S)$, and skew $\text{Skew}_\alpha$:

$$
\sigma_0 S^{\beta_S} = \sigma_{ATM}(S)
$$

$$
\beta_S + \frac{\beta_V}{\sigma_0} = \frac{2 \text{Skew}_\alpha}{\sigma_{ATM}(S)} \equiv \Lambda
$$

The first equation is known as the backbone that defines the trajectory of the ATM volatility given a change in the spot price:

$$
\frac{\sigma_{ATM}(S) - \sigma_{ATM}(S_0)}{\sigma_{ATM}(S_0)} \approx \beta_S \frac{S - S_0}{S_0}
$$

(2)
Model implied stickiness and volatility regimes
If we insist on model-inconsistent delta (change in spot with volatility level unchanged):
fit backbone beta $\beta_S$ to reproduce specified stickiness ratio
adjust $\beta_V$ so that the model fits the market skew

Using stickiness ratio $R(t_n)$ along with (2), we obtain that empirically:

$$\beta_S(t_n) = \frac{\text{Skew}_\alpha(t_{n-1})}{\sigma_{ATM}(t_{n-1})} R(t_n)$$

Thus, given an estimated value of the stickiness rate we imply $\beta_S$

Finally, by mixing parameters $\beta_S$ and $\beta_V$ we can produce different volatility regimes:
sticky-delta with $\beta_S = 0$ and $\beta_V \approx 2\text{Skew}_\alpha$
sticky-local volatility with $\beta_V = 0$ and $\beta_S \approx \Lambda$

From the empirical data we infer that, approximately,
$\beta_S \approx 70\% \Lambda$ and $\beta_V = 30\% \times 2\text{Skew}_\alpha$
Beta stochastic volatility model
Let me consider pure SV beta expressed in terms of normalized volatility factor $Y(t)$ (this version is applied in practice for beta SV with local volatility):

$$\frac{dS(t)}{S(t)} = (1 + Y(t))\sigma dW(t), \ S(0) = S_0$$

$$dY(t) = \tilde{\beta}_V \frac{dS(t)}{S(t)}, \ Y(0) = 0$$

(3)

where $\sigma$ is the overall level of the volatility (can be deterministic or local $\sigma(t, S)$)
$Y(t)$ is the normalized volatility factor fluctuation around zero

Volatility parameter $\tilde{\beta}_V$ can be implied from short term ATM volatility $\sigma_{ATM}$ and skew Skew$_{\alpha}$:

$$\tilde{\beta}_V = \frac{2\text{Skew}_{\alpha}}{\sigma_{ATM}}$$

(4)

The goal now is to investigate the dynamics of the skew
Beta stochastic volatility model. Skew

Inverting the above equation:

\[ \text{Skew}(t) = \frac{1}{2} \tilde{\beta}_V \sigma_{ATM}(t) \]  \hspace{1cm} (5)

Dynamically, using (4):

\[
d\text{Skew}(t) = \frac{1}{2} \tilde{\beta}_V d\sigma_{ATM}(t) \propto \frac{1}{2} \tilde{\beta}_V \sigma_{ATM}(t) dY(t) \\
\propto \frac{1}{2} \sigma_{ATM}(t) \left( \tilde{\beta}_V \right)^2 \frac{dS(t)}{S(t)}
\]

To test the above equation empirically, we apply the regression model for coefficient \( q \):

\[
\text{Skew}(t_n) - \text{Skew}(t_{n-1}) = q \left[ 2 \left( \text{Skew}(t_{n-1}) \right)^2 \frac{1}{\sigma_{ATM}(t_{n-1})} \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})} \right]
\]  \hspace{1cm} (6)

First, we test (5)
Skew vs ATM volatility (crisis) for 1m and 1y ATM vols

1m Skew vs ATM vol, crisis period Oct 07 - Mar 09

\[ y = -0.2452x - 0.584 \]

\[ R^2 = 0.1023 \]

1y Skew vs ATM vol, crisis period Oct 07 - Mar 09

\[ y = -0.0147x - 0.2473 \]

\[ R^2 = 0.0052 \]
Skew vs ATM volatility (recovery) for 1m and 1y ATM vols

1m Skew vs ATM vol, recovery period Mar 09 - Feb 11

$y = -0.005x - 0.6058$

$R^2 = 6E-06$

1y Skew vs ATM vol, recovery period Mar 09 - Feb 11

$y = 0.2671x - 0.3333$

$R^2 = 0.1335$
Skew vs ATM volatility (range) for 1m and 1y ATM vols

$y = -1.2322x - 0.4895$
$R^2 = 0.5236$

$y = -0.0968x - 0.2942$
$R^2 = 0.0347$
Skew vs ATM volatility. Summary

<table>
<thead>
<tr>
<th></th>
<th>crisis</th>
<th>recovery</th>
<th>range-bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skew-vol beta 1m</td>
<td>-0.25</td>
<td>-0.01</td>
<td>-1.23</td>
</tr>
<tr>
<td>Skew-vol beta 1y</td>
<td>-0.01</td>
<td>-0.27</td>
<td>-0.10</td>
</tr>
<tr>
<td>$R^2$ 1m</td>
<td>10%</td>
<td>0%</td>
<td>52%</td>
</tr>
<tr>
<td>$R^2$ 1y</td>
<td>1%</td>
<td>13%</td>
<td>3%</td>
</tr>
</tbody>
</table>

Empirically, in general, a high level of ATM volatility implies a higher level of the skew but the relationship is not strong and is mixed.

For short-term skew, the relationship is stronger in crisis and range-bound periods.

For longer-term skew, the relationship is stronger in recovery periods.

Next, we test (6) for relationship between changes in the skew and spot returns.
Skew vs price return (crisis) for 1m and 1y skews
Skew vs price return (recovery) for 1m and 1y skews
Skew vs price return (range) for 1m and 1y skews
Skew vs price return. Summary

<table>
<thead>
<tr>
<th></th>
<th>crisis</th>
<th>recovery</th>
<th>range-bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skew-return beta 1m</td>
<td>-0.14</td>
<td>0.37</td>
<td>0.21</td>
</tr>
<tr>
<td>Skew-return beta 1y</td>
<td>-0.16</td>
<td>-0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>$R^2$ 1m</td>
<td>6%</td>
<td>18%</td>
<td>13%</td>
</tr>
<tr>
<td>$R^2$ 1y</td>
<td>6%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

The short-term skew appears to be somewhat dependent on spot changes:
during crisis periods, negative returns decrease the skew (de-leveraging reduces need for downside protection)
during recovery and range-bound periods, negative returns increase the skew (risk-aversion is high especially during recovery period)

The long-term skew does not appear to depend on spot returns
Skew vs price return. Conclusions

The skew does not seem to depend on either volatility or spot dynamics (especially for longer maturities)

About 20% of variations in the short-term skew can be attributed to changes in the spot

Only jumps appear to have a reasonable explanation for the skew (the fear of a crash does not (or little) depend on current values of variables)
Full beta stochastic volatility model I

The pricing version of the beta model is specified as follows:

\[
\frac{dS(t)}{S(t)} = \mu(t)dt + (1 + Y(t))\sigma dW^{(0)}(t), \quad S(0) = S
\]
\[
dY(t) = -\kappa Y(t)dt + \beta_V(1 + Y(t))\sigma dW^{(0)}(t) + \varepsilon dW^{(1)}(t), \quad Y(0) = 0
\]

where:
\begin{itemize}
  \item $\beta_V$ ($\beta_V < 0$) is the rate of change in the volatility corresponding to change in the spot price
  \item $\varepsilon$ is idiosyncratic volatility of volatility
  \item $\kappa$ is the mean-reversion rate
  \item $W^{(0)}(t)$ and $W^{(1)}(t)$ are two Brownians with $dW^{(0)}(t)dW^{(1)}(t) = 0$
  \item $\mu(t)$ is the risk-neutral drift
\end{itemize}

$\sigma$ is the overall level of volatility

$\sigma$ is set to either constant volatility $\sigma_{CV}$ or deterministic volatility $\sigma_{DV}(t)$, or local stochastic volatility $\sigma_{LSV}(t, S)$

$\sigma = \{\sigma_{CV}, \sigma_{DV}(t), \sigma_{LSV}(t, S)\}$
Beta stochastic volatility model. II

Using dynamics (7), for the log-spot, $X(t) = \ln \left( \frac{S(t)}{S(0)} \right)$ we obtain:

\[
dX(t) = \mu(t)dt - \frac{1}{2}\sigma^2(1 + Y(t))^2 dt + \sigma(1 + Y(t))dW(0)(t), \quad X(0) = 0
\]

\[
dY(t) = \beta\sigma (1 + Y(t))dW(0)(t) - \kappa Y(t)dt + \varepsilon dW(1)(t), \quad Y(0) = 0
\]

(8)

with

\[
dY(t)dY(t) = \left( \varepsilon^2 + \beta^2\sigma^2(1 + Y(t))^2 \right) dt
\]

\[
dX(t)dY(t) = \beta\sigma^2(1 + Y(t))^2 dt
\]

The pricing equation for value function $U(t, T, X, Y)$ has the form:

\[
U_t + \frac{1}{2}\sigma^2(1 + 2Y + Y^2) \left[ U_{XX} - U_X \right] + \mu(t)U_X
\]

\[
+ \frac{1}{2} \left( \varepsilon^2 + \beta^2\sigma^2 \left( 1 + 2Y + Y^2 \right) \right) U_{YY} - \kappa Y U_Y
\]

\[
+ \beta\sigma^2 \left( 1 + 2Y + Y^2 \right) U_{XY} - r(t)U = 0
\]

(9)

where $r(t)$ is the discount rate and subscripts denote partial derivatives.
**Beta stochastic volatility model. III**
The parameters of the stochastic volatility, $\beta$, $\varepsilon$ and $\kappa$ are specified before the calibration.

We calibrate the local volatility $\sigma \equiv \sigma_{LSV}(t, S)$, using either a parametric local volatility (CEV) or non-parametric local volatility, so that the vanilla surface is matched by construction.

For calibration of $\sigma_{LSV}(t, S)$ we apply the conditional expectation (Lipton A, The vol smile problem, *Risk*, February 2002):

$$\sigma_{LSV}^2(T, K)\mathbb{E}\left[(1 + Y(T))^2 \mid S(T) = K\right] = \sigma_{LV}^2(T, K)$$

where $\sigma_{LV}^2(T, K)$ is the local Dupire volatility.

The above expectation is computed by solving the forward PDE corresponding to pricing PDE (9) using finite-difference methods and computing $\sigma_{LSV}^2(T, K)$ stepping forward in time.

Once $\sigma_{LSV}(t, S)$ is calibrated we use either backward PDE-s or MC simulation for valuation of exotic options.
Beta SV model. Approximation for call price I

I propose an affine approximation for pricing equation (9) with constant or deterministic volatility $\sigma$:

$$G(t, T, X, Y; \Phi) = \exp \left\{ -\Phi X + A^{(0)} + A^{(1)}Y + A^{(2)}Y^2 \right\}$$

with $A^{(n)}(T; T) = 0$, $n = 0, 1, 2$

By substitution this into PDE (9) and collecting terms proportional to $Y$ and $Y^2$ only, we obtain a system of ODE-s for $A^{(n)}(t)$:

$$A_t^{(0)} + v_0A^{(2)} + \frac{1}{2}v_0(A^{(1)})^2 - \Phi A^{(1)}c_0 + \frac{1}{2}q = 0$$

$$A_t^{(1)} + \frac{1}{2}v_1(A^{(1)})^2 + 2v_0A^{(1)}A^{(2)} + v_1A^{(2)} - \kappa A^{(1)} - \Phi \left( 2c_0A^{(2)} + c_1A^{(1)} \right) + q = 0$$

$$A_t^{(2)} + \frac{1}{2}v_2(A^{(1)})^2 + 2v_0(A^{(2)})^2 + 2v_1A^{(1)}A^{(2)} + v_2A^{(2)} - 2\kappa A^{(2)} - \Phi \left( 2c_1A^{(2)} + c_2A^{(1)} \right) + \frac{1}{2}q = 0$$

where

$q = \sigma^2 (\Phi^2 + \Phi)$, $v_0 = \varepsilon^2 + \beta^2\sigma^2$, $v_1 = 2\beta^2\sigma^2$, $v_2 = \beta^2\sigma^2$, $c_0 = \beta\sigma^2$, $c_1 = 2\beta\sigma^2$, $c_2 = \beta\sigma^2$

This is system is solved by means of Runge-Kutta methods.

It is straightforward to incorporate time-dependent model parameters (but not space-dependent local volatility $\sigma_{LSV}(t, X)$)
Beta SV model. Approximation for call price II

As a result, for pricing vanilla options, we can apply the standard methods based on the Fourier inversion.

The value of the call option with strike $K$ is computed by applying Lipton-Lewis formula:

$$C(t, T, S, Y) = e^{-\int_t^T r(t')dt'} \left( e^{\int_t^T \mu(t')dt'} S - \frac{K}{\pi} \int_0^\infty \Re \left[ \frac{G(t, T, x, Y; ik - 1/2)}{k^2 + 1/4} \right] dk \right)$$

where $x = \ln(S/K) + \int_t^T \mu(t')dt'$
### Beta SV model. Approximation for call price III

Implied model volatilities computed by approximation formula vs numerical PDE using \( \beta = -7.63, \ \varepsilon = 0.35, \ \kappa = 4.32 \) and constant volatilities: \( \sigma_{CV}(1m) = 19.14\%, \ \sigma_{CV}(1y) = 23.96\%, \ \sigma_{CV}(2y) = 24.21\% \)
Properties of the volatility process, assuming constant vol $\sigma_{CV}$

Instantaneous variance of $Y(t)$ is given by:

$$dY(t)dY(t) = \left(\beta^2\sigma_{CV}^2(1 + Y(t))^2 + \varepsilon^2\right)dt$$

which has systemic part proportional to $Y(t)$ and idiosyncratic part $\varepsilon$

In a stress regime, for large values of $Y(t)$, the variance is dominated by $\beta^2\sigma_{CV}^2Y^2(t)$ (close to a log-normal model for volatility process)

The volatility process has **steady-state variance** (so that the volatility approaches stationary distribution in the long run):

$$\mathbb{E}[Y^2(t) \mid Y(0) = 0] = \frac{\varepsilon^2 + \beta^2\sigma_{CV}^2}{2\kappa - \beta^2\sigma_{CV}^2} \left(1 - e^{-(2\kappa - \beta^2\sigma_{CV})t}\right)$$

**Effective mean-reversion for** the volatility of variance is:

$$2\kappa - \beta^2\sigma_{CV}^2$$

**Steady state variance** of volatility is

$$\frac{\varepsilon^2 + \beta^2\sigma_{CV}^2}{2\kappa - \beta^2\sigma_{CV}^2}$$
**Instantaneous correlation** between $dY(t)$ and $dX(t)$:

$$
\rho(dX(t)dY(t)) = \frac{\beta \sigma^2_{CV}(1 + Y(t))^2}{\sqrt{(\varepsilon^2 + \beta^2 \sigma^2_{CV}(1 + Y(t))^2)} \sqrt{\sigma^2_{CV}(1 + Y(t))^2}}
$$

With high volatility $Y(t)$ is large so letting $Y(t) \to \infty$ we obtain that

$$
\rho(dX(t)dY(t)) \big|_{Y(t) \approx \infty} = -1
$$

In a normal regime, $Y(t) \approx 0$, so that obtain:

$$
\rho(dX(t)dY(t)) \big|_{Y(t) \approx 0} = -\frac{1}{\sqrt{\left(\frac{\varepsilon^2}{\beta^2 \sigma^2_{CV}} + 1\right)}}
$$

The beta SV model introduces **state-dependent spot-volatility correlation**, with high volatility leading to absolute negative correlation.

In contrast, Heston and Ornstein-Uhlenbeck based SV models always assume constant instantaneous correlation.
The steady state density

Steady state density function $G(Y)$ of volatility factor $Y(t)$ in dynamics (7) solves the following equation:

$$\frac{1}{2} \left[ \left( \varepsilon^2 + \beta^2 \sigma_{CV}^2 \left( 1 + 2Y + Y^2 \right) \right) G \right]_{YY} + [\kappa Y G]_Y = 0$$

We can show that $G(Y)$ exhibits the power-like behavior for large values of $Y$:

$$\lim_{Y \to +\infty} G(Y) = Y^{-\alpha}, \quad \alpha = 2 \left( 1 + \frac{\kappa}{(\beta \sigma_{CV})^2} \right)$$

This power-like behavior contrasts with Heston and exponential volatility models which imply exponential tails for the steady-state density of the volatility.

Thus, the beta SV model predicts higher probabilities of large values of instantaneous volatility.
The steady state density. Tails

Tails in the beta SV model (green) vs Ornstein-Uhlenbeck (OU) exponential SV model (red) with $\alpha = 5$ in beta SV model and equivalent vol-of-vol in OU SV model $\epsilon_{OU} = 0.61$
Case Study I: Correlation skew

Apply experiment:
1) Compute implied volatilities from options on the index (say, S&P500)
2) Using implied volatilities (probability density function) of stocks in this index, and stock-stock Gaussian correlations, compute option prices on the index, compute the implied volatility from these prices

Empirical observation (correlation skew):
The index skew computed in 1) is steeper than that computed in 2)  

Explanation:
Stocks become strongly correlated during big sell-offs
Index skew reflects premium for buying puts on a basket of stocks

Modelling approach:
Correlation skew cannot be replicated using Gaussian correlation
Stochastic and/or local correlations can be applied
But only **SV model with jumps can produce realistic dynamics and reproduce the correlation skew**

Next we augment the beta SV model with jumps and apply it to reproduce the correlation skew
**SPY and select sector ETF-s**

Consider:

SPY - the ETF tracking the S&P500 index

Select sector ETF-s - ETF-s tracking 9 sectors of the S&P500 index

<table>
<thead>
<tr>
<th>ETF</th>
<th>SPY weight</th>
<th>Sector</th>
</tr>
</thead>
<tbody>
<tr>
<td>XLK</td>
<td>20.46%</td>
<td>INFORMATION TECHNOLOGY</td>
</tr>
<tr>
<td>XLF</td>
<td>14.88%</td>
<td>FINANCIALS</td>
</tr>
<tr>
<td>XLV</td>
<td>12.38%</td>
<td>HEALTH CARE</td>
</tr>
<tr>
<td>XLP</td>
<td>11.66%</td>
<td>CONSUMER STAPLES</td>
</tr>
<tr>
<td>XLY</td>
<td>11.31%</td>
<td>CONSUMER DISCRETIONARY</td>
</tr>
<tr>
<td>XLE</td>
<td>11.15%</td>
<td>ENERGY</td>
</tr>
<tr>
<td>XLI</td>
<td>10.81%</td>
<td>INDUSTRIALS</td>
</tr>
<tr>
<td>XLU</td>
<td>3.84%</td>
<td>UTILITIES</td>
</tr>
<tr>
<td>XLB</td>
<td>3.51%</td>
<td>MATERIALS</td>
</tr>
</tbody>
</table>
Index and sector vols and skews
Term structure of ATM volatilities (left) and 105%-95% skews (right)
SPY ATM vol (black line) can be viewed as a weighted average of sector ATM vols
SPY skew (black line) is steeper than weighted average skews of sectors
Calibration of the beta SV model I

First, calibrate the beta SV model with constant SV parameters. Calibration is based on intuition and experience with the model.

**Volatility beta**, $\beta$, is set by

$$\beta = \frac{\sigma_{IMP}(6m, 5\%) - \sigma_{IMP}(6m, -5\%)}{0.05\sigma_{IMP}(6m, 0\%)} = \frac{2\text{Skew}_{5\%}(6m)}{\sigma_{ATM}(6m)}$$

where $\sigma_{IMP}(6m, k\%)$ is 6m implied vol for forward-based log-strike $k$.

**Idiosyncratic volatility** $\varepsilon$ is set according to:

$$\varepsilon^2 = \sigma_{IMP}^2(6m, 0\%)\beta^2\frac{1 - (\rho^*)^2}{(\rho^*)^2}$$

$\rho^*$ is spot-vol correlation for 6m vol implied by SV model with Orstein-Uhlenbeck process for SV driver.

**Reversion speed** $\kappa$ is adjusted to fit term structure of 1y-3y 105% – 95% skew.

Term structure of model level vols $\sigma_{DV}(t)$ are calibrated by construction (by root search) so that the ATM implied vol is fitted exactly.
Calibration of the beta SV model II

Calibrated parameters of the beta SV model to SPY and sector ETFs

<table>
<thead>
<tr>
<th></th>
<th>SPY</th>
<th>XLK</th>
<th>XLF</th>
<th>XLV</th>
<th>XLP</th>
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<th>XLE</th>
<th>XLI</th>
<th>XLU</th>
<th>XLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>-4.77</td>
<td>-3.72</td>
<td>-2.78</td>
<td>-5.34</td>
<td>-5.31</td>
<td>-3.93</td>
<td>-2.75</td>
<td>-3.06</td>
<td>-5.55</td>
<td>-2.89</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0.40</td>
<td>0.39</td>
<td>0.67</td>
<td>0.35</td>
<td>0.33</td>
<td>0.42</td>
<td>0.55</td>
<td>0.41</td>
<td>0.38</td>
<td>0.39</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>1.45</td>
<td>1.60</td>
<td>1.30</td>
<td>1.25</td>
<td>1.15</td>
<td>1.40</td>
<td>1.40</td>
<td>1.30</td>
<td>1.25</td>
<td>1.45</td>
</tr>
<tr>
<td>$\rho^*$</td>
<td>-0.81</td>
<td>-0.78</td>
<td>-0.60</td>
<td>-0.80</td>
<td>-0.80</td>
<td>-0.76</td>
<td>-0.66</td>
<td>-0.72</td>
<td>-0.79</td>
<td>-0.73</td>
</tr>
</tbody>
</table>

Next we illustrate plots of the term structure of market and model implied 105% – 95% skew and 1y implied vols accross range of strikes

Typically, if the beta SV model is fits 105% – 95% skew, then it will fit the skew accross different strikes

Beta SV model is similar to one-factor SV models - the model fits well longer-term skews (above one-year) while it is unable to fit short-term skews (up to one year) unless beta parameter $\beta$ is large

By actual pricing, small discrepancies in implied vols are eliminated by local vol part
Calibrated parameters

Volatility Beta

Idiosyncratic vol-of-vol

Reversion Speed

Spot-Vol Corr

SPY  XLK  XLF  XLV  XLP  XLY  XLE  XLI  XLU  XLB

SPY  XLK  XLF  XLV  XLP  XLY  XLE  XLI  XLU  XLB

SPY  XLK  XLF  XLV  XLP  XLY  XLE  XLI  XLU  XLB

SPY  XLK  XLF  XLV  XLP  XLY  XLE  XLI  XLU  XLB

65
XLK - information technology

![Graph showing skew and implied vol for XLK information technology]

**Skew**
- XLK, Market Skew
- XLK, Model Skew

**Implied Vol**
- 1y Implied vols
- XLK, Market impl vol
- XLK, Model impl vol
XLF - financials

![Graph showing skew and implied volatilities for XLF]

- Skew:
  - XLF, Market Skew
  - XLF, Model Skew

- Implied Volatilities:
  - XLF, Market Impl Vol
  - XLF, Model Impl Vol
XLV - health care

![Graph showing skew and implied vol for XLV, Market Skew and XLV, Model Skew.]

![Graph showing 1y implied vols for XLV, Market impl vol and XLV, Model impl vol.]

69
XLP - consumer staples

![Graph showing skew and implied volatility over time for XLP. The skew increases over time, and the implied volatility decreases.]
XLY - consumer discretionary

![Graph showing skew and implied vol over time]

- XLY, Market Skew
- XLY, Model Skew

![Graph showing 1y implied vols]

- XLY, Market impl vol
- XLY, Model impl vol
XLE - energy

![Graph showing skew and implied vol for XLE]

- XLE, Market Skew
- XLE, Model Skew

![Graph showing 1y implied vol]

- XLE, Market impl vol
- XLE, Model impl vol
XLI - industrials

![Graph showing skew and implied volatility for XLI industrials.]

- **Skew**
  - XLI, Market Skew
  - XLI, Model Skew

- **1y Implied Vols**
  - XLI, Market Impl Vol
  - XLI, Model Impl Vol

**Dimensions:**
- XLI Industrials
- Skew T
- Impl Vol
- 1y Implied Vols

**Outputs:**
- 3m
- 6m
- 9m
- 12m
- 18m
- 21m
- 24m
- 27m
- 30m
- 33m
- 36m

**Values:**
- 15%
- 25%
- 35%
- 5%
- 15%
- 50%
- 60%
- 70%
- 80%
- 90%
- 100%
- 110%
- 120%
- 130%
- 140%
- 150%

**K**

**73**
XLU - utilities

![Diagram 1: XLU, Market Skew vs Time](image1)

![Diagram 2: XLU, Model Skew vs Implied Vol](image2)
XLB - materials

![Graph showing skew over time]

- XLB, Market Skew
- XLB, Model Skew

![Graph showing 1y implied vol vs. strike]

- XLB, Market impl vol
- XLB, Model impl vol
Multi-asset beta SV model

The SV model without jumps cannot reproduce the correlation skew

We present multi-asset beta SV model with simultaneous jumps in assets and their volatilities:

\[
\frac{dS_n(t)}{S_n(t)} = \mu_n(t)dt + (1 + Y_n(t))\sigma_n dW_n(t) + (e^{\nu_n} - 1)(dN(t) - \lambda dt)
\]
\[
dY_n(t) = -\kappa_n Y_n(t)dt + \beta_n (1 + Y_n(t))\sigma_n dW_n(t) + \varepsilon_n dW^{(1)}(t) + \eta_n dN(t)
\]

where \(n = 1, \ldots, N\)

\(dW_n(t)\) are Brownians for asset prices with specified correlation matrix
\(W^{(1)}(t)\) is the joint driver for idiosyncratic volatilities
\(N(t)\) is the joint Poisson process with intensity \(\lambda\) for simultaneous shocks in prices and volatilities

\(\nu_n, \nu_n < 0\), are constant jump amplitudes in log-price
\(\eta_n, \eta_n > 0\), are constant jump amplitudes in volatilities
Jump calibration
Based on my presentation for Global Derivatives in Paris, 2011
The idea is based on linear impact of jumps on the short-term implied skew:

\[ \sigma_{imp}(K) \approx \sigma - \frac{\lambda \nu}{\sigma} \ln \left( \frac{S}{K} \right) \]

Specify \( w_{jd} \) - the percentage of the skew attributed to jumps
Set jump intensity as follows:

\[ \lambda = \frac{\left( \text{Skew}_{5\%}(1y) \right)^2}{w_{jd}} \]

The jump size is implied as follows:

\[ \nu = -\sqrt{w_{jd}\sigma_{ATM}(1y)} \sqrt{\lambda} = \frac{w_{jd}\sigma_{ATM}(1y)}{\text{Skew}_{5\%}(1y)} \]

Jump size in volatility, \( \eta \), can be calibrated to options on the VIX skew or options on the realized variance
Empirically, \( \eta \approx 2 \)
Jump calibration for ETF. I
Set \( w_{jd} = 50\% \) and imply jump intensity \( \lambda_{SPY} \) from 1y 5% skew for SPY ETF

**Individual jump sizes** are set using \( \lambda_{SPY} \) and sector specific ATM volatility \( \sigma_{ATM,n}(1y) \):

\[
\nu_n = -\frac{\sqrt{w_{jd}}\sigma_{ATM,n}(1y)}{\sqrt{\lambda_{SPY}}}
\]

**Jump size in volatility**, \( \eta \), is set uniformly \( \eta = 2 \) (realized jump in ATM volatility will be proportional to ATM volatility of sector ETF)

Previously specified \( \beta \), \( \varepsilon \) and \( \kappa \) are reduced by 25%

Next we illustrate plots of term structure of market and model implied 105% – 95% skew and 1y implied volatilities accross range of strikes

Beta SV model with jumps produces steep forward skews for short maturities and is consistent with term structure of skew

Again, by actual pricing, small discrepancies in implied vols are eliminated by local vol part
Jump calibration for ETF. II

For recent data, $\lambda = 0.17$ and jump sizes are shown in table and figure.

<table>
<thead>
<tr>
<th>SPY</th>
<th>XLK</th>
<th>XLF</th>
<th>XLV</th>
<th>XLP</th>
<th>XLY</th>
<th>XLE</th>
<th>XLI</th>
<th>XLU</th>
<th>XLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.32</td>
<td>-0.37</td>
<td>-0.35</td>
<td>-0.28</td>
<td>-0.26</td>
<td>-0.35</td>
<td>-0.41</td>
<td>-0.37</td>
<td>-0.26</td>
<td>-0.41</td>
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![Graph showing jump sizes for different ETFs]
SPY

![Skew Graph]

- SPY, Market Skew
- SPY, Model Skew

![Implied Vol Graph]

- SPY, Market impl vol
- SPY, Model impl vol

K

1y Implied vols

10% 20% 30% 40%

50% 60% 70% 80% 90% 100% 110% 120% 130% 140% 150%
XLK - information technology

![Chart 1: Skew](chart1.png)

![Chart 2: 1y Implied Vols](chart2.png)
XLF - financials
XLV - health care

[Graph showing skew across different time periods (3m, 6m, 9m, 12m, 15m, 18m, 21m, 24m, 27m, 30m, 33m, 36m). The graph compares XLV, Market Skew and XLV, Model Skew.

[Graph showing 1y implied vol distribution with different strike prices (K)]
XLP - consumer staples
XLY - consumer discretionary

![Graph showing skew and implied vol for XLY, Market Skew and XLY, Model Skew over different time periods (3m to 36m). The skew graph has a y-axis from -0.5 to 0.0 and an x-axis from 3m to 36m. The implied vol graph has a y-axis from 10% to 40% and an x-axis from 50% to 150% with corresponding vol levels.]
XLE - energy

![Chart showing skew and implied volatility for XLE, with market skew and model skew plotted against time. The chart also displays 1-year implied volatilities with strike prices ranging from 50% to 150%.]
XLI - industrials
XLU - utilities

![Graph showing skew over time and implied volatility over strike]

- Skew values for XLU, Market Skew and XLU, Model Skew are plotted against time (3m, 6m, 9m, 12m, 15m, 18m, 21m, 24m, 27m, 30m, 33m, 36m).
- Implied volatility values for XLU, Market impl vol and XLU, Model impl vol are plotted against strike (5%, 10%, 15%, 20%, 25%, 30%, 35%, 40%, 45%, 50%, 55%, 60%, 65%, 70%, 75%, 80%, 85%, 90%, 95%, 100%, 105%, 110%, 115%, 120%, 125%, 130%, 135%, 140%, 145%, 150%).
XLB - materials
Correlation matrix for sector ETF-s
Sector-wise correlation are estimated using time series

<table>
<thead>
<tr>
<th></th>
<th>XLK</th>
<th>XLF</th>
<th>XLV</th>
<th>XLP</th>
<th>XLY</th>
<th>XLE</th>
<th>XLI</th>
<th>XLU</th>
<th>XLB</th>
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<td>88%</td>
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Correlation skew. Illustration
Next we compare the index skew implied from:
1) SPY options (SPY)
2) basket of ETF priced using local volatility with Gaussian correlations (Local Vol)
3) basket of ETF priced using beta SV model (Beta LSV)
4) basket of ETF priced using beta SV model with jumps (Beta LSV+Jumps)
Correlation skew. Conclusion

Local volatility and stochastic volatility models without jumps cannot reproduce the correlation skew

Only jumps can introduce the correlation skew in a robust way

In my example, I calibrated jumps to 1y skew so the model fits 1y correlation skew, but model correlation skew flattens for 2y (problem with data for long-dated ETF options?)

Perhaps more elaborate jump process is necessary (probably though spot- and volatility-dependent intensity process)
Case Study II: Conditional forward skew

We imply forward volatility by computing forward-start option conditional that $S(\tau)$ starts in range $(D - \Delta, D + \Delta)$, typically $\Delta = 5\%$, with the following pay-off:

$$1_{\{D - \Delta < S(\tau) < D + \Delta\}} \left( \frac{S(T)}{S(\tau)} - K \right)^+$$

In the BSM model, the variables $S(\tau)$ and $\frac{S(T)}{S(\tau)}$ are independent so that the BSM value of this pay-off is the probability of $S(\tau)$ hitting the range times the value of the forward start call

Under alternative models, we compute the above expectation, $PV$, by means of MC simulations and in addition compute the hitting probability, $P$, $P = \mathbb{E} \left[ 1_{\{D - \Delta < S(\tau) < D + \Delta\}} \right]$

Then we imply the conditional volatility using the BSM inversion for call with strike $K$, time to maturity $T - \tau$, and value $PV/P$

We compare two models: local volatility (LV) and beta SV with local vol (LSV)
Conditional forward skew II

6m6m conditional skew for $D = 110\%$ (left) and $D = 90\%$ (right)
Conditional forward skew III

6m6m conditional skew for $D = 125\%$ (left) and $D = 75\%$ (right)
6m6m conditional skew for $D = 140\%$ (left) and $D = 60\%$ (right)
Conclusions
I presented the beta stochastic volatility model that

1) Has intuitive parameters (volatility beta) that can be explained using empirical data

2) Calibration of parameters for SV process and jumps is straightfor-ward and intuitive (no non-linear optimization methods are necessary)

3) Allows to mix parameters to reproduce different regimes of volatility and the equity skew

4) Equipped with jumps, allows to reproduce correlation skew for multi-underlyings

5) Produces very steep forward skews

6) The driver for the instantaneous volatility has nice properties: fat tails and level dependent spot-volatility correlations
Open questions
1) Better understanding of relationship between model parameters of market observables (ATM vol, skew, their term structures)

2) Model implied risk incorporating the stickiness ratio

3) Numerical methods (numerical PDE, analytic approximations)

4) Calibration of jumps and correlation skew

5) Illustrate/prove that only SV model with jumps is consistent with observed empirical features:
   A) Stickiness ratio is between 1 and 2
   B) Steep correlation skew

Models with local volatility and correlation may be consistent with A) and B) but they are not consistent with observed dynamics thus producing wrong hedges

Thank you for your attention!
Disclaimer

The opinions and views expressed in this presentation are those of the author alone and do not necessarily reflect the views and policies of Bank of America Merrill Lynch.