Risk, Return, and Ross Recovery

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Reminding Your $\mathbb{P}$'s & $\mathbb{Q}$'s: FTAP#1

- Depending on the context, the "First Fundamental Theorem of Asset Pricing" is either a proven mathematical theorem or a folk theorem that relates two probability measures usually denoted $\mathbb{P}$ and $\mathbb{Q}$.
- Here we regard $\mathbb{P}$ as indicating not only which states or paths can occur, but also indicating the frequency with which the market believes they occur. We assume that this frequency is reflected in market prices, which also reflect attitudes towards risk.
- FTAP#1 takes $\mathbb{P}$ as given along with an arbitrage-free market containing a money market account $S_{0t} > 0, t \geq 0$. These assumptions imply the existence of a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for each asset's spot price $S_{it} \in \mathbb{R}, i = 0, 1 \ldots, n$, the ratio $\frac{S_{it}}{S_{0t}}$ is a $\mathbb{Q}$ martingale.
- The probability measure $\mathbb{Q}$ is referred to as an "equivalent martingale measure" (EMM).

Reminding Your $\mathbb{P}$'s & $\mathbb{Q}$'s: FTAP#2

- Loosely, a financial market is said to be "complete" if the payoff of any contingent claim in a specified set can be replicated by dynamic trading in a specified subset of the "basis assets".
- FTAP#2 says that when FTAP#1 holds, a market is complete if and only if the EMM $\mathbb{Q}$ is unique.

Overview of this Talk

- There are four parts to this talk:
  1. Ross Recovery Theorem for Finite State Markov Chains
  2. Review of John Long's Numeraire Portfolio
  3. Ross Recovery for Bounded Diffusions
  4. Ross Recovery for Unbounded Diffusions
- The operating assumptions will be different in each section. Within a section, only one set of assumptions holds.

Part I: $\mathbb{P}$, $\mathbb{Q}$, and Ross Recovery

- Recall that the real-world probability measure $\mathbb{P}$ quantifies the market's belief about the frequencies of future states or paths. Suppose that $\mathbb{P}$ is ex ante unknown.
- Assume that the market is complete and that in contrast to $\mathbb{P}$, the risk-neutral probability measure $\mathbb{Q}$ is known ex ante.
- In 2011, Steve Ross began circulating a working paper called "The Recovery Theorem" whose first theorem gives sufficient conditions under which knowing $\mathbb{Q}$ implies knowing $\mathbb{P}$ exactly.
- We call his Theorem 1 the Ross Recovery Theorem.

The Ross Recovery Theorem

- Theorem 1 in Ross (2011) states that:
  1. if markets are complete, and
  2. if the utility function of the representative investor is state-independent and intertemporally additively separable with a constant rate of time preference and:
  3. if there is a single state variable X which under $\mathbb{Q}$ is a time homogeneous Markov chain with a finite number of states,
- then under $\mathbb{P}$, X is also a finite state Markov chain and one can recover the real-world transition probability matrix P of X from the assumed known and unique risk-neutral transition probability matrix Q.

Vas Ist Das?

- A couple of us at Morgan Stanley were intrigued by Ross's conclusion that $\mathbb{P}$ can be obtained from $\mathbb{Q}$, but we wondered whether it was necessary to restrict preferences. Is there a preference-free way to obtain $\mathbb{P}$ from $\mathbb{Q}$?
- We also wondered whether it was necessary that the state variable X be a finite state Markov chain. The industry practice is to use models with a continuous state space. We wondered in particular if $\mathbb{P}$ can be obtained from $\mathbb{Q}$ when X is a diffusion under $\mathbb{Q}$.
- By restricting the $\mathbb{Q}$ dynamics of something called the numeraire portfolio, we showed that there is a preference-free way to obtain $\mathbb{P}$ from $\mathbb{Q}$ when the state variable X is a diffusion on a bounded domain.
- We are presently working to extend the results to a diffusion on an unbounded domain, which is a much harder problem.

Part 2: The Numeraire Portfolio

- In 1990, John Long introduced a notion which he called the *numeraire portfolio*.
- A numeraire is any self-financing portfolio whose price is alway positive.
- Long showed that if any set of assets is arbitrage-free, then there always exists a self-financing portfolio of them whose value is always positive.
- Furthermore, if the price of each asset is expressed relative to the value of this numeraire portfolio, then the relative price is a $\mathbb{P}$ martingale.
- Long's discovery of the existence of the numeraire portfolio allows one to do arbitrage-free pricing of derivative securities without using the probability measure $\mathbb{Q}$. Instead, one deflates every asset price by the value L of Long's numeraire portfolio, and then uses the real-world probability measure $\mathbb{P}$ as the martingale measure.

Intuition on the Numeraire Portfolio

- The important mathematical property of the Money Market Account (MMA) is that its price process has sample paths of bounded variation.
- When the MMA is used to finance all purchases and denominate all gains, then the $\mathbb{P}$ expected return on each asset position is usually not zero and in equilibrium would usually be ascribed to risk premium.
- The important mathematical property of the numeraire portfolio is that its value process is allowed to have sample paths of *unbounded* variation.
- One can use this extra flexibility to construct a portfolio whose positive value covaries with each asset price so that when gains are expressed in units of the numeraire portfolio, the mean gain under $\mathbb{P}$ on each asset is zero.
- If a particular asset is thought to have a positive risk premium, then one constructs the numeraire portfolio to covary positively with this asset's price. When gains on the asset are parked in the numeraire portfolio, this positive covariation kills the risk premium.

Intuition for Positive Continuous Spot Prices

- Let S_{0t} be the spot price of the MMA and suppose that we have n risky assets with spot prices S_i for $i = 1, \ldots, n$.
- Long's results hold when these spot prices are arbitrary semi-martingales, but let's try to gain intuition by restricting each security's spot price S_i to be positive and continuous over time.
- By definition, the value L of Long's numeraire portfolio is also positive and in this case, its sample paths would also be continuous over time.
- Applying Itô's formula to the ratio S_{it}/L_t, we have:

$$\frac{d(S_{it}/L_t)}{S_{it}/L_t} = \frac{dS_{it}}{S_{it}} - \frac{dL_t}{L_t} - \frac{dS_{it}}{S_{it}}\frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2, \qquad i = 1 \ldots n,$$

where $\frac{dS_{it}}{S_{it}}\frac{dL_t}{L_t} \equiv \frac{d\langle S_i, L\rangle_t}{S_{it}L_t}$ and $(\frac{dL_t}{L_t})^2 \equiv \frac{d\langle L\rangle_t}{L_t^2}$.

Intuition for Positive Continuous Spot Prices (Con'd)

- Recall that for $i = 1, \ldots, n$, the ratio of each spot price S_i to the value L of Long's numeraire portfolio has $\mathbb{P}$ dynamics:

$$\frac{d(S_{it}/L_t)}{S_{it}/L_t} = \frac{dS_{it}}{S_{it}} - \frac{dL_t}{L_t} - \frac{dS_{it}}{S_{it}}\frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2.$$

- Suppose that the $\mathbb{P}$ dynamics for each risky spot price S_i are:

$$dS_{it}/S_{it} = (r_t + \pi_{it})dt + \sigma_{it} dB_{it}, \qquad i = 1, \ldots, n,$$

where π_i is the i-th asset's risk premium, σ_i is the i-th asset's volatility process, and B_i is a driving SBM.

- Let π_{Lt} and σ_{Lt} be the risk premium and volatility process for L:

$$dL_t/L_t = (r_t + \pi_{Lt})dt + \sigma_{Lt} dB_{Lt}.$$

- Taking expectations of the top equation under $\mathbb{P}$, we get:

$$E_t^{\mathbb{P}} \frac{d(S_{it}/L_t)}{S_{it}/L_t} = \pi_{it} - \pi_{Lt} - \sigma_{iL,t} + \sigma_{Lt}^2, \qquad i = 1 \ldots n.$$

Intuition for Positive Continuous Spot Prices (Con'd)

- Recall that the mean relative return under $\mathbb{P}$ for each risky asset is given by:

$$E_t^{\mathbb{P}} \frac{d(S_{it}/L_t)}{S_{it}/L_t} = \pi_{it} - \pi_{Lt} - \sigma_{iL,t} + \sigma_{Lt}^2, \qquad i = 1 \dots n,$$

  where the π's denote risk premia, $\sigma_{iL,t}dt \equiv \frac{dS_{it}}{S_{it}} \frac{dL_t}{L_t}$, while $\sigma_{Lt}^2 dt \equiv (dL_t/L_t)^2$.

- Suppose we construct L so that $\sigma_{iL,t} = \pi_{it}$ for $i = 1 \dots n$.
- Since L is just the value of a portfolio of these assets, we have that $\sigma_{L,t}^2 = \pi_{Lt}$ and hence the mean relative return on each asset vanishes!
- It follows that the real world expected return on each asset would be:

$$E_t^{\mathbb{P}} \frac{dS_{it}}{S_{it}} = r_t dt + \frac{dS_{it}}{S_{it}} \frac{dL_t}{L_t} \qquad i = 1 \dots n.$$

- If we can determine the covariation of S with L under $\mathbb{Q}$, then this covariation is unchanged when we switch to $\mathbb{P}$, and hence we can determine the instantaneous risk premium $\pi_{it} \equiv E_t^{\mathbb{P}} \frac{dS_{it}}{S_{it}} - r_t dt$ on each asset.

Risk and Return for the Numeraire Portfolio

- Recall that the risk premium of the numeraire portfolio IS its instantaneous variance rate.

$$E_t^{\mathbb{P}} \frac{dL_t}{L_t} - r_t dt = \left(\frac{dL_t}{L_t}\right)^2 = \sigma_{Lt}^2 dt, \qquad t \geq 0.$$

- One could not imagine a simpler relation between risk premium and risk.
- Since $\pi_{Lt} = \sigma_{Lt}^2$, the SDE for L under $\mathbb{P}$ is:

$$\frac{dL_t}{L_t} = (r_t + \sigma_{Lt}^2)dt + \sigma_{Lt} dB_{Lt} = r_t dt + \sigma_{Lt}(dB_{Lt} + \sigma_{Lt} dt), \qquad t \geq 0.$$

- It follows that the market price of the Brownian risk dB_{Lt} is σ_{Lt}.
- Our goal is now to determine σ_L, which represents both risk and reward.

Part 3: Ross Recovery for Bounded Diffusions

- Using Long's numeraire portfolio, we replace Ross's restrictions on the form of preferences with our restrictions on how prices evolve under $\mathbb{Q}$.
- More precisely, we suppose that the prices of some given set of assets are all driven by a univariate time-homogenous bounded diffusion process, X.
- Letting L denote the value of the numeraire portfolio for these assets, we furthermore assume that L is also driven by X and t and that (X, L) is a bivariate time homogenous diffusion.
- We show that these assumptions determine the real-world dynamics of all assets in the given set.

Our Assumptions

- We assume no arbitrage for some finite set of assets which includes a money market account (MMA).
- As a result, there exists a risk-neutral measure $\mathbb{Q}$ under which spot prices deflated by the MMA balance evolve as martingales.
- We assume that under $\mathbb{Q}$, the driver X is a time homogeneous *bounded* diffusion process:

$$dX_t = b(X_t)dt + a(X_t)dW_t, \qquad t \in [0, T],$$

where $X_t \in (\ell, u), t \geq 0$, $a(x) > 0$, and where W is SBM under $\mathbb{Q}$.

- We also assume that under $\mathbb{Q}$, the value L of the numeraire portfolio solves:

$$\frac{dL_t}{L_t} = r(X_t)dt + \sigma_L(X_t)dW_t, \qquad t \in [0, T].$$

- We know the functions $b(x)$, $a(x)$, and $r(x)$ but not $\sigma_L(x)$. How to find it?

Value Function of the Numeraire Portfolio

- Recalling that X is our driver, we assume:

$$L_t \equiv L(X_t, t), \qquad t \in [0, T],$$

where $L(x,t)$ is a positive function of $x \in \mathbb{R}$ and time $t \in [0, T]$.

- Applying Itô's formula, the volatility of L is:

$$\sigma_L(x) \equiv \frac{1}{L(x,t)} \frac{\partial}{\partial x} L(x,t) a(x) = a(x) \frac{\partial}{\partial x} \ln L(x,t).$$

- Dividing by $a(x) > 0$ and integrating w.r.t. x:

$\ln L(x,t) = \int^x \frac{\sigma_L(y)}{a(y)} dy + f(t)$, where $f(t)$ is the constant of integration.

- Exponentiating implies that the value of the numeraire portfolio separates multiplicatively into a positive function $\pi(\cdot)$ of the driver X and a positive function $p(\cdot)$ of time t:

$$L(x,t) = \pi(x) p(t),$$

where: $\pi(x) = e^{\int^x \frac{\sigma_L(y)}{a(y)} dy}$ and $p(t) = e^{f(t)}$.

Separation of Variables

- The numeraire portfolio value function $L(x,t)$ must solve the following linear parabolic PDE to be self-financing:

$$\frac{\partial}{\partial t}L(x,t) + \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}L(x,t) + b(x)\frac{\partial}{\partial x}L(x,t) = r(x)L(x,t).$$

- On the other hand, the last slide shows that this value separates as:

$$L(x,t) = \pi(x)p(t).$$

- Using Bernoulli's classical separation of variables argument, we know that:

$$p(t) = p(0)e^{\lambda t},$$

and that:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in [\ell, u].$$

Regular Sturm Liouville Problem

- Recall the ODE on the last slide:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in [\ell, u].$$

  where $\pi(x)$ and λ are unknown.

- Whichever boundary conditions we are allowed to impose, they will be separated. As a result, we have a *regular* Sturm Liouville problem.
- From Sturm Liouville theory, we know that there exists an eigenvalue λ_0, smaller than all of the other eigenvalues, and an associated positive eigenfunction, $\pi_0(x)$, which is unique up to positive scaling.
- All of the eigenfunctions associated to the other eigenvalues switch signs at least once.
- One can numerically solve for both the smallest eigenvalue λ_0 and its associated positive eigenfunction, $\pi_0(x)$. The positive eigenfunction $\pi_0(x)$ is unique up to positive scaling.

Value Function of the Numeraire Portfolio

- Recall that λ_0 is the known lowest eigenvalue and $\pi_0(x)$ is the associated eigenfunction, positive and known up to a positive scale factor.
- Knowing λ_0 and knowing $\pi_0(x)$ up to a positive constant implies that we also know the value function of the numeraire portfolio up to a positive constant, since:

$$L(x,t) = \pi_0(x)e^{\lambda_0 t}, \qquad x \in [\ell, u], t \in [0, T].$$

- As a result, the volatility of the numeraire portfolio is *uniquely* determined:

$$\sigma_L(x) = a(x)\frac{\partial}{\partial x} \ln \pi_0(x), \qquad x \in [\ell, u].$$

- Mission accomplished! Let's see what the market believes.

Real World Dynamics of the Numeraire Portfolio

- Recall that in our diffusion setting, Long (1990) showed that the *real world* dynamics of L are given by:

$$\frac{dL_t}{L_t} = [r(X_t) + \sigma_L^2(X_t)]dt + \sigma_L(X_t)dB_t, \qquad t \geq 0,$$

  where B is a standard Brownian motion under the real-world probability measure $\mathbb{P}$.

- In equilibrium, the risk premium of the numeraire portfolio is simply $\sigma_L^2(x)$.
- Since we have determined $\sigma_L(x)$ on the last slide, the risk premium of the numeraire portfolio has also been uniquely determined.
- The market price of Brownian risk is simply $\sigma_L(X_t)$. The function $\sigma_L(x)$ is now known, but what about the dynamics of X?

Real World Dynamics of the Driver

- From Girsanov's theorem, the dynamics of the driver X under the real-world probability measure $\mathbb{P}$ are:

$$dX_t = [b(X_t) + \sigma_L(X_t)a(X_t)]dt + a(X_t)dB_t, \qquad t \geq 0,$$

  where recall B is a standard Brownian motion under $\mathbb{P}$.

- Hence, we now know the real world dynamics of the driver X.
- We still have to determine the real-world transition density of the driver X.

Real World Transition PDF of the Driver

- From the change of numeraire theorem, the Radon Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{-\int_0^T r(X_t)dt} \frac{L(X_T, T)}{L(X_0, 0)} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{\lambda_0 T} e^{-\int_0^T r(X_t)dt},$$

  since $L(x,t) = \pi_0(x)e^{\lambda_0 t}$.

- Solving for the real-world PDF $d\mathbb{P}$ gives:

$$d\mathbb{P} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{-\lambda_0 T} e^{-\int_0^T r(X_t)dt} d\mathbb{Q} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{-\lambda_0 T} d\mathbb{A}.$$

  where $d\mathbb{A} \equiv e^{-\int_0^T r(X_t)dt} d\mathbb{Q}$ denotes the Arrow Debreu state pricing density.

- Knowing $d\mathbb{Q}$ implies that we also know the Arrow Debreu state pricing density $d\mathbb{A}$, at least numerically. As we also know the positive function $\frac{\pi_0(y)}{\pi_0(x)}$ and the positive function $e^{-\lambda T}$, we know $d\mathbb{P}$, the real-world transition PDF of X.

Real-World Dynamics of Spot Prices

- Also from Girsanov's theorem, the dynamics of the $i-$th spot price S_{it} under $\mathbb{P}$ are uniquely determined as:

$$dS_{it} = [r(X_t)S_i(X_t,t)+\sigma_L(X_t)\frac{\partial}{\partial x}S_i(X_t,t)a^2(X_t)]dt+\frac{\partial}{\partial x}S_i(X_t,t)a(X_t)dB_t,$$

  where for $x \in (\ell, u), t \in [0, T]$, $S_i(x,t)$ solves the following linear PDE:

$$\frac{\partial}{\partial t}S_i(x,t) + \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}S_i(x,t) + b(x)\frac{\partial}{\partial x}S_i(x,t) = r(x)S_i(x,t),$$

  subject to appropriate boundary and terminal conditions. If $S_{it} > 0$, then the SDE at the top can be expressed as:

$$\frac{dS_{it}}{S_{it}} = r(X_t)+\sigma_L(X_t)\frac{\partial}{\partial x}\ln S_i(X_t,t)a^2(X_t)]dt+\frac{\partial}{\partial x}\ln S_i(X_t,t)a(X_t)dB_t, t \geq 0.$$

- In equilibrium, the instantaneous risk premium is just $d\langle \ln S_i, \ln L\rangle_t$, i.e. the increment of the quadratic covariation of returns on S_i with returns on L.

Part 4: Examples of Diffusions on Unbounded Domain

- Our results thus far apply only to diffusions on bounded domains.
- Hence, our results thus far can't be used to determine whether one can uniquely determine $\mathbb{P}$ in models like Black Scholes (1973) and Cox Ingersoll & Ross (1985) (aka CIR), where the diffusing state variable X lives on an unbounded domain such as $(0, \infty)$.
- We don't yet know the general theory here, but we do know two interesting examples of it.

Example 1: Black Scholes Model for a Stock Price

- Suppose that the state variable X is a stock price whose initial value is observed to be the positive constant S_0.
- Suppose we assume or observe zero interest rates and dividends and we assume that the spot price is geometric Brownian motion under $\mathbb{Q}$:

$$\frac{dX_t}{X_t} = \sigma dW_t, \qquad t \geq 0.$$

- Suppose only one stock option trades and from its observed market price, we learn the numerical value of σ.
- All of our previous assumptions are in place except now we have allowed the diffusing state variable X to live on the unbounded domain $(0, \infty)$.

Example 1: Black Scholes Model (con'd)

- Recall the general ODE governing the positive function $\pi(x)$ & the scalar λ:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u).$$

- In the BS model with zero rates, $a^2(x) = \sigma^2 x^2$, $b(x) = r(x) = 0$, $\ell = 0$, and $u = \infty$ so we want a positive function $\pi(x)$ and a scalar λ solving the Euler ODE:

$$\frac{\sigma^2 x^2}{2}\pi''(x) = -\lambda\pi(x), \qquad x \in (0, \infty).$$

- In the class of twice differentiable functions, there are an uncountably infinite number of eigenpairs (λ, π) with π positive. This result implies Ross can't recover here because there are too many candidates for the value of the numeraire portfolio.
- However, all of the positive functions $\pi(x)$ are not square integrable. If we insist on this condition as well, then there are no candidates for the value of the numeraire portfolio. Ross can't recover again, but for a different reason.

Example 2: CIR Model for the Short Rate

- Suppose that after calibrating to caps, floors and swaptions, we find that the short interest rate r solves the following mean-reverting square root process under $\mathbb{Q}$:

$$dr_t = (\mu - \kappa r_t)dt + \sigma\sqrt{r_t}dW_t, \qquad t \geq 0.$$

  where r_0, μ, κ, and σ are all known positive constants.

- Recall the general ODE governing the positive function $\pi(x)$ & the scalar λ:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u).$$

- Here, $a^2(x) = \sigma^2 x$, $b(x) = \mu - \kappa x$, $r(x) = x$, $\ell = 0$, and $u = \infty$ so we want a positive function $\pi(x)$ and a scalar λ solving the linear ODE:

$$\frac{\sigma^2 x}{2}\pi''(x) + (\mu - \kappa x)\pi'(x) - x\pi(x) = -\lambda\pi(x), \qquad x \in (0, \infty).$$

Example 2: CIR Model (Con'd)

- Recall we want a positive function $\pi(x)$ and a scalar λ solving the ODE:

$$\frac{\sigma^2 x}{2}\pi''(x) + (\mu - \kappa x)\pi'(x) - x\pi(x) = -\lambda\pi(x), \qquad x \in (0,\infty).$$

- The scale density $s(x)$ and speed density $m(x)$ of the CIR process are:

$$s(x) = x^{\frac{-2\mu}{\sigma^2}} e^{\frac{2\kappa}{\sigma^2}x} \qquad m(x) = \frac{2}{\sigma^2} x^{\frac{2\mu}{\sigma^2}-1} e^{\frac{-2\kappa}{\sigma^2}x}.$$

- These densities are used to determine the nature of the boundaries 0 and ∞. As is well known, ∞ is a natural boundary, while 0 is an entrance boundary if $\mu \geq \frac{\sigma^2}{2}$ and regular if $\mu \in (0, \frac{\sigma^2}{2})$. When 0 is an entrance boundary, we must have the reflecting boundary condition $\lim_{x\downarrow 0} \frac{f'(x)}{s(x)} = 0$. When 0 is a regular boundary, we choose to have this condition apply.
- Suppose we consider the weighted square integrable function space $L^2((0,\infty), m(x)dx)$ Then the spectrum is discrete with eigenvalues and eigenfunctions known in closed form (see e.g. Gorovoi and Linetsky (2004)).

Example 2: CIR Model (Con'd)

- Examining the closed form expression for the eigenvalues, we observe that the lowest eigenvalue is $\lambda_0 = \frac{\mu}{\sigma^2}(\gamma - \kappa)$, where $\gamma \equiv \sqrt{\kappa^2 + 2\sigma^2}$.
- Examining the closed form expression for the eigenfunctions, we observe that the associated eigenfunction is $\pi_0(x) = e^{-\frac{\gamma-\kappa}{\sigma^2}x}$, which is positive. All of the other eigenfunctions switch signs at least once.
- It follows that in the CIR short rate model, the value function for the numeraire portfolio must be:

$$L(r,t) = \pi_0(r)e^{\lambda_0 t} = e^{-\frac{\gamma-\kappa}{\sigma^2}r + \frac{\mu}{\sigma^2}(\gamma-\kappa)t}, \qquad r > 0, t \geq 0.$$

- Under $\mathbb{P}$, Girsanov's theorem implies that the short rate r solves:

$$dr_t = (\mu - \gamma r_t)dt + \sigma\sqrt{r_t}dB_t, \qquad t \geq 0,$$

  where B is a standard Brownian motion. Hence, the short rate is still a CIR process under $\mathbb{P}$, but with larger mean reversion since $\gamma > \kappa$.
- In this example, Ross recovery succeeded and moreover we used his model!

Summary

- We highlighted Ross's Theorem 1 and proposed an alternative preference-free way to derive the same financial conclusion.
- Our approach is based on imposing time homogeneity on the $\mathbb{Q}$ dynamics of the value L of Long's numeraire portfolio when it is driven by a bounded time homogeneous diffusion process X.
- We showed how separation of variables allows us to separate beliefs from preferences.
- We explored two examples of unbounded diffusions. In the first (Black Scholes model for stock price), we are unable to recover the real world drift of the stock. In the second (CIR model for the short rate), we were able to recover the real world dynamics of the short rate.
- At present, we do not have a general theory giving sufficient conditions for when Ross recovery succeeds for unbounded diffusions.

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