# Optimal execution with mulitplicative price impact

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- We consider an investor who has a large number of stock shares to sell within a given time frame. Rapid selling of the stock may depress the stock price, while slicing the big order into many smaller blocks of orders to be executed sequentially over time may take too long to realise. Such an investor is therefore faced with the problem of how to slice the order, when to trade and at what price, etc.

- The study of the optimal execution problem was initiated by Bertsimas and Lo (1998) who analysed a discrete random walk model and by Almgren and Chriss (1999, 2001) and Almgren (2003) who considered continuous time Bachelier-type models. An incomplete list of notable contributions includes Huberman and Stanzl (2004), He and Mamaysky (2005), Obizhaeva and Wang (2012), Almgren and Lorenz (2007), Engle and Ferstenberg (2007), Schied and Schöneborn (2009), Alfonsi and Schied (2010), Alfonsi, Fruth and Schied (2010), Schied, Schöneborn and Tehranchi (2010), Predoiu, Shaikhet and Shreve (2011), and Løkka (2012).

- Modelling stock prices by an arithmetic Brownian motion with additive impact of large stock sales is a common feature in the references above. Optimal strategies turn out to be more or less static or deterministic, which leads to predictable trading patterns.

- Gatheral and Schied (2011) studied a continuous time Black & Scholes-type model with additive price impact. Bertsimas and Lo (1998) and Bertsimas, Lo and Hummel (1999) considered discrete time models with multiplicative price impact. Forsyth, Kennedy, Tse and Windcliff (2012, 2013) proposed a continuous time Black and Scholes-type model with multiplicative price impact.

- We denote by $Y_t$ the total number of shares held by the investor at time $t$. Also, we denote by $\xi^{\mathrm{s}}_t$ (resp., $\xi^{\mathrm{b}}_t$) the total number of shares that the investor has sold (resp., bought) up to time $t$, so that

$$Y_t = y - \xi^{\mathrm{s}}_t + \xi^{\mathrm{b}}_t \tag{1}$$

where $y \geq 0$ is the number of shares held by the investor at time 0.

- We allow for no short-selling: $Y_t \geq 0$ for all $t \geq 0$.

- A primary aim of the investor is to liquidate all share holdings by a time horizon $\overline{T} \in (0, \infty]$. We therefore consider only trading strategies $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$ such that

$$Y_{\overline{T}+} = 0, \text{ if } \overline{T} < \infty, \quad \text{and} \quad \lim_{T \to \infty} Y_T = 0, \text{ if } \overline{T} = \infty. \tag{2}$$

- In the absence of any transaction from the investor, the stock price is modelled by the geometric Brownian motion
$$dX_t^0 = \mu X_t^0 \, dt + \sigma X_t^0 \, dW_t, \quad X_0^0 = x > 0, \tag{3}$$
where $\mu$ and $\sigma \neq 0$ are given constants.

- We assume that small transactions made by the investor affect the share price proportionally to its value. In particular, if the investor sells (resp., buys) a small amount $\varepsilon > 0$ of shares at time $t$, then the share price exhibits a jump of size
$$\Delta X_t = X_{t+} - X_t = -\lambda \varepsilon X_t \quad \big(\text{resp., } \Delta X_t = X_{t+} - X_t = \lambda \varepsilon X_t\big), \tag{4}$$
for some constant $\lambda > 0$. A *small* sale (resp., buy) of size $\varepsilon > 0$ is associated with the expressions
$$X_{t+} = (1 - \lambda\varepsilon) X_t \simeq e^{-\lambda\varepsilon} X_t \quad \big(\text{resp., } X_{t+} = (1 + \lambda\varepsilon) X_t \simeq e^{\lambda\varepsilon} X_t\big). \tag{5}$$
If we view the sale of $\Delta\xi_t^{\mathrm{s}}$ shares as $N$ individual sales of $\varepsilon = \Delta\xi_t^{\mathrm{s}}/N$ shares each, then, for $N$ large enough, we obtain
$$X_{t+} = e^{-\lambda N \varepsilon} X_t = e^{-\lambda \Delta\xi_t^{\mathrm{s}}} X_t. \tag{6}$$
Similarly, we can see that buying $\Delta\xi_t^{\mathrm{b}}$ shares is associated with the jump $X_{t+} = e^{\lambda \Delta\xi_t^{\mathrm{b}}} X_t$.

- In view of the above considerations, we model the stock price dynamics by the stochastic equation

$$dX_t = \mu X_t \, dt - \lambda X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} + \lambda X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} + \sigma X_t \, dW_t, \tag{7}$$

  where

$$X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} = X_t \, d(\xi^{\mathrm{s}})_t^{\mathrm{c}} + \frac{1}{\lambda} X_t \big[1 - e^{-\lambda \Delta \xi_t^{\mathrm{s}}}\big] = X_t \, d(\xi^{\mathrm{s}})_t^{\mathrm{c}} + X_t \int_0^{\Delta \xi_t^{\mathrm{s}}} e^{-\lambda u} \, du \tag{8}$$

  and

$$X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} = X_t \, d(\xi^{\mathrm{b}})_t^{\mathrm{c}} + \frac{1}{\lambda} X_t \big[e^{\lambda \Delta \xi_t^{\mathrm{b}}} - 1\big] = X_t \, d(\xi^{\mathrm{b}})_t^{\mathrm{c}} + X_t \int_0^{\Delta \xi_t^{\mathrm{b}}} e^{\lambda u} \, du, \tag{9}$$

  where the process $(\xi^{\mathrm{s}})^{\mathrm{c}}$ (resp., $(\xi^{\mathrm{b}})^{\mathrm{c}}$) is the continuous part of the process $\xi^{\mathrm{s}}$ (resp., $\xi^{\mathrm{b}}$).

- Using Itô's formula, we can verify that the solution to (7) is given by

$$X_t = x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t - \lambda \xi_t^{\mathrm{s}} + \lambda \xi_t^{\mathrm{b}} + \sigma W_t\right) = X_t^0 \exp\left(-\lambda \xi_t^{\mathrm{s}} + \lambda \xi_t^{\mathrm{b}}\right) \tag{10}$$

  where $X^0$ is the geometric Brownian motion given by (3).

- In this model, transactions made by the investor have a permanent impact. There are several extensions of the model that can accommodate transient impact. For instance, we can replace the dynamics given by (10) by $X_t = X_t^0 e^{Z_t}$, where

$$Z_t = -\lambda \int_{[0,t[} G(t-s)\, d\big(d\xi_t^{\mathrm{s}} - d\xi_t^{\mathrm{b}}\big),$$

for some kernel $G$. In this context, if we choose $G(t-s) = e^{-\gamma t} e^{\gamma s}$, for some constant $\gamma > 0$, then

$$dZ_t = -\gamma Z_t\, dt - \lambda\, d\xi_t^{\mathrm{s}} + \lambda\, d\xi_t^{\mathrm{b}}.$$

In any case, the resulting optimisation problem's state space involves four variables (namely, $t$, $x$, $y$ and $z$) instead of three (namely, $t$, $x$ and $y$).

We leave this as well as other extensions accommodating resilience of the stock price for future research.

- If we consider the sale of $\Delta\xi_t^{\mathrm{s}}$ shares at time $t$ as equivalent to the sale of $N$ packets of shares of small size $\varepsilon = \Delta\xi_t^{\mathrm{s}}/N$, then we can see that such a sale should result in a revenue of

$$\sum_{j=0}^{N-1} e^{-\lambda j\varepsilon} X_t \varepsilon \simeq \int_0^{\Delta\xi_t^{\mathrm{s}}} X_t e^{-\lambda u}\, du = \frac{1}{\lambda} X_t \big[1 - e^{-\lambda \Delta\xi_t^{\mathrm{s}}}\big]. \tag{11}$$

  In view of this observation and a similar one concerning the buying of $\Delta\xi_t^{\mathrm{b}}$ shares at time $t$, we associate the performance criterion

$$I_{\overline{T},x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) = \begin{cases} J_{\overline{T},x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}), & \text{if } \overline{T} < \infty, \\ \limsup_{T\to\infty} J_{T,x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}), & \text{if } \overline{T} = \infty, \end{cases} \tag{12}$$

  with each liquidation strategy $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$, where $J_{T,x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$ is defined by

$$J_{T,x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) = \mathbb{E}\left[\int_{[0,T]} e^{-\delta t}\big[X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} - C_{\mathrm{s}}\, d\xi_t^{\mathrm{s}} - C_{\mathrm{b}}\, d\xi_t^{\mathrm{b}}\big]\right], \tag{13}$$

  for $(T, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+$.

  This performance criterion is the expected revenue one featuring in the models studied, e.g., by Bersimas and Lo (1998) and Gatheral (2010). Alternative performance indices give rise to several variants of the model that could be the subject of future research.

- The investor's objective is to maximise $I_{\overline{T},x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}})$ over all liquidation strategies $(\xi^{\mathrm{s}},\xi^{\mathrm{b}})$. Accordingly, we define the problem's value function $v$ by

$$v(\overline{T},x,y) = \sup_{(\xi^{\mathrm{s}},\xi^{\mathrm{b}})\in\mathcal{A}_{\overline{T},y}} I_{\overline{T},x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}). \tag{14}$$

  where $\mathcal{A}_{\overline{T},y}$ is the family of all admissible strategies.

- **Assumption.** $\mu$, $\sigma \neq 0$, $\delta \geq \max\{\mu, 0\}$, $C_{\mathrm{s}}, C_{\mathrm{b}} \geq 0$ and $\kappa = \lambda > 0$ are constants.

- **Definition.** An admissible round-trip trade with time horizon $T \in \mathbb{R}_+^*$ is any pair $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ of $(\mathcal{F}_t)$-adapted increasing càglàd processes such that $\zeta_0^{\mathrm{s}} = \zeta_0^{\mathrm{b}} = 0$,

$$\zeta_{T+}^{\mathrm{s}} = \zeta_{T+}^{\mathrm{b}} \quad \text{and} \quad \sup_{t \in [0,T]} \left( \zeta_{t+}^{\mathrm{s}} - \zeta_{t+}^{\mathrm{b}} \right) \leq \Gamma, \tag{15}$$

for some constant $\Gamma > 0$, which may depend on the trading strategy itself.

- **Definition.** The market allows for arbitrage opportunities if there exists a round-trip trade with resulting revenue that is positive and strictly positive with strictly positive probability, namely, if there exists a round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ such that

$$R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) = \int_{[0,T]} \left[ X_t \circ_{\mathrm{s}}^{\lambda} d\zeta_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}}^{\kappa} d\zeta_t^{\mathrm{b}} - C_{\mathrm{s}}\, d\zeta_t^{\mathrm{s}} - X_t\, d\zeta_t^{\mathrm{b}} \right] \geq 0 \tag{16}$$

and $\mathbb{P}\left( R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) > 0 \right) > 0$.

- **Definition.** A price manipulation is a round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ resulting in a strictly positive expected revenue, namely, $\mathbb{E}\left[ R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) \right] > 0$, where $R$ is defined by (16). An unbounded price manipulation is a sequence of round-trip trades $(\zeta^{\mathrm{s},n}, \zeta^{\mathrm{b},n})$ such that $\lim_{n \to \infty} \mathbb{E}\left[ R(\zeta^{\mathrm{s},n}, \zeta^{\mathrm{b},n}) \right] = \infty$.

- Notice that in these definitions $\delta = 0$ !

- If we allow for asymmetric impact of buying and selling, namely, if we model the stock price dynamics by

$$dX_t = \mu X_t \, dt - \lambda X_t \circ_{\mathrm{s}}^{\lambda} d\xi_t^{\mathrm{s}} + \kappa X_t \circ_{\mathrm{b}}^{\kappa} d\xi_t^{\mathrm{b}} + \sigma X_t \, dW_t, \tag{17}$$

  where $\circ_{\mathrm{s}}^{\lambda}$ (resp., $\circ_{\mathrm{b}}^{\kappa}$) is defined by (8) (resp., (9) with $\kappa$ in place of $\lambda$), for some $\kappa \neq \lambda$, then the market may present arbitrage opportunities

  If $\kappa > \lambda > 0$, then the market presents arbitrage opportunities and arbitrarily high risk-free profits can be realised by simple round-trip strategies that buy an appropriate number of shares at time 0 and then sell them at an future appropriate time.

  If $\lambda > \kappa > 0$, then the market may present arbitrage opportunities, which can be realised by a simple round-trip strategy that short-sells an appropriate number of shares at time 0 and then buys them back at an future appropriate time.

- Suppose that we are in the context of the model we study, namely,

$$dX_t = \mu X_t \, dt - \lambda X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} + \lambda X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} + \sigma X_t \, dW_t. \tag{18}$$

  The market does not allow for arbitrage opportunities.

  If $\mu < 0$, then price manipulation may exist.

  If $\mu > 0$, then unbounded price manipulation exists.

  If $\mu = 0$, then there exists no price manipulation.

- Consider the optimal execution problem formulated above.
- If $0 \leq \delta < \mu$ *in violation* of our assumption, then round-trip trades involving no short-selling can realise arbitrarily high expected payoffs and

$$v(\overline{T}, x, y) = \infty \quad \text{for all } (\overline{T}, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+. \tag{19}$$

- The optimal liquidation strategy involves no buying of shares, namely,

$$v(\overline{T}, x, y) = \sup_{\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T}, y}} I_{\overline{T}, x, y}(\xi^{\mathrm{s}}, 0). \tag{20}$$

In particular, the market does not allow for transaction-triggered price manipulation.

- The value function satisfies

$$\frac{1}{\lambda}x\left[1-e^{-\lambda y}\right]-C_{\mathrm{s}}y\leq v(\overline{T},x,y)\leq\frac{1}{\lambda}x\left[1-e^{-\lambda y}\right] \tag{21}$$

for all $\overline{T}\in(0,\infty]$ and $(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+$.

- If $C_{\mathrm{s}}=0$, then it is optimal to sell all shares at time 0 and the value function is given by

$$v(\overline{T},x,y)=\frac{1}{\lambda}x\left[1-e^{-\lambda y}\right]\quad\text{for all }\overline{T}\in(0,\infty]\text{ and }(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+. \tag{22}$$

- Suppose that $\delta=\mu\geq 0$. If $\overline{T}\in\mathbb{R}_+^*$, then it is optimal to sell all available shares at $\overline{T}$. On the other hand, if $\overline{T}=\infty$, then selling all available shares at time $n=1,2,\ldots$ provides a sequence of $\varepsilon$-optimal strategies. In this case, the value function is given by

$$v(\overline{T},x,y)=\begin{cases}\frac{1}{\lambda}x\left[1-e^{-\lambda y}\right]-e^{-\delta\overline{T}}C_{\mathrm{s}}y, & \text{if }\overline{T}\in\mathbb{R}_+^*,\\ \frac{1}{\lambda}x\left[1-e^{-\lambda y}\right], & \text{if }\overline{T}=\infty,\end{cases} \tag{23}$$

for all $(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+$.

- Suppose that $\overline{T} < \infty$ and $\delta > \max\{\mu, 0\}$.
- We expect that the value function $v$ of the stochastic control problem identifies with an appropriate solution $w : \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ to the HJB equation

$$\max\Big\{ -w_t(t,x,y) + \mathcal{L}w(t,x,y),\ -\lambda x w_x(t,x,y) - w_y(t,x,y) + x - C_\mathrm{s} \Big\} = 0, \tag{24}$$

with boundary condition

$$w(0,x,y) = \frac{1}{\lambda} x \big[1 - e^{-\lambda y}\big] - C_\mathrm{s} y, \tag{25}$$

where

$$\mathcal{L}w(t,x,y) = \frac{1}{2}\sigma^2 x^2 w_{xx}(t,x,y) + \mu x w_x(t,x,y) - \delta w(t,x,y). \tag{26}$$

Also, we define

$$\mathcal{W} = \big\{(t,x,y) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ -w_t(t,x,y) + \mathcal{L}w(t,x,y) = 0\big\}, \tag{27}$$

$$\mathcal{S} = \big\{(t,x,y) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ \lambda x w_x(t,x,y) + w_y(t,x,y) - x + C_\mathrm{s} = 0\big\}. \tag{28}$$

- Suppose that, at a given time, the investor's horizon is $t > 0$, the share price is $x > 0$ and the investor holds an amount $y > 0$ of shares. At that time, the investor is faced with two possible actions. The first one is to wait for a short time $\Delta t$ and then continue optimally:

$$v(t, x, y) \geq \mathbb{E}\left[e^{-\delta \Delta t} v\big(t - \Delta t, X_{\Delta t}, y\big)\right].$$

Applying Itô's formula and dividing by $\Delta t$ before letting $\Delta t \downarrow 0$, we obtain

$$-v_t(t, x, y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x, y) + \mu x v_x(t, x, y) - \delta v(x, y) \leq 0. \tag{29}$$

The second possibility is to sell a small amount $\varepsilon > 0$ of shares, and then continue optimally:

$$v(t, x, y) \geq v(t, x - \lambda x \varepsilon, y - \varepsilon) + (x - C_{\mathrm{s}})\varepsilon.$$

Rearranging terms and letting $\varepsilon \downarrow 0$, we obtain

$$-\lambda x v_x(t, x, y) - v_y(t, x, y) + x - C_{\mathrm{s}} \leq 0. \tag{30}$$

The Markovian character of the problem implies that one of these three possibilities should be optimal and one of (29)–(30) should hold with equality at any point in the state space.

- Given a time horizon $\overline{T} \in (0, \infty)$, suppose that a function $w : [0, \overline{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ is a $C^{1,2,1}$ solution to the HJB equation (24)–(25) such that

$$-C_{\mathrm{s}} y \leq w(t, x, y) \leq \frac{1}{\lambda} x \quad \text{for all } (t, x, y) \in [0, \overline{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+. \tag{31}$$

If, for all initial conditions $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, there exists $\xi^{\mathrm{s}\star} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}$ such that

$$(X_t^\star, Y_t^\star) \in \mathcal{W} \quad \text{for all } t \geq 0,\ \mathbb{P}\text{-a.s.}, \tag{32}$$

$$\xi^{\mathrm{s}\star}_{t+} = \int_{[0,t]} \mathbf{1}_{\{(X_t^\star, Y_t^\star) \in \mathcal{S}\}} \, d\xi^{\mathrm{s}\star}_t \quad \text{for all } t \geq 0,\ \mathbb{P}\text{-a.s.}, \tag{33}$$

where $X^\star$ and $Y^\star$ are the share price and shares held processes associated with the liquidation strategy $(\xi^{\mathrm{s}\star}, 0)$, then $w$ identifies with the value function $v$ of the stochastic control problem. In particular,

$$v(\overline{T}, x, y) = \sup_{\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}} J_{\overline{T},x,y}(\xi^{\mathrm{s}}, 0) = w(\overline{T}, x, y) \quad \text{for all } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+, \tag{34}$$

and $(\xi^{\mathrm{s}\star}, 0)$ is an optimal liquidation strategy.

- Suppose now that $\overline{T} = \infty$ and write $v(x, y)$ instead of $v(\infty, x, y)$.

  We assume in what follows that

$$\mu < \delta \quad \text{and} \quad C_{\mathrm{s}} > 0. \tag{35}$$

- We solve the problem that arises by constructing an appropriate solution $w : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ to the HJB equation

$$\max \left\{ \mathcal{L}w(x, y),\ -\lambda x w_x(x, y) - w_y(x, y) + x - C_{\mathrm{s}} \right\} = 0, \tag{36}$$

  where $\mathcal{L}$ is defined by (26), with boundary condition

$$w(x, 0) = 0 \quad \text{for all } x > 0. \tag{37}$$

- We look for a solution $w$ to (36)–(37) that is characterised by a function $F : \mathbb{R}_+ \to \mathbb{R}_+$ that partitions the state space $\mathbb{R}_+^* \times \mathbb{R}_+$ into two regions, the "waiting" region $\mathcal{W}$ and the "selling" region $\mathcal{S}$, defined by

$$\mathcal{W} = \left\{ (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ y > 0 \text{ and } x < F(y) \right\} \cup \left( \mathbb{R}_+^* \times \{0\} \right), \tag{38}$$

$$\mathcal{S} = \left\{ (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ y > 0 \text{ and } x \geq F(y) \right\}. \tag{39}$$

- Inside $\mathcal{W}$, $w$ should satisfy the differential equation

$$\frac{1}{2}\sigma^2 x^2 w_{xx}(x,y) + \mu x w_x(x,y) - \delta w(x,y) = 0.$$

The only solution to this ODE that remains bounded as $x \downarrow 0$ is given by

$$w(x,y) = A(y)x^n, \tag{40}$$

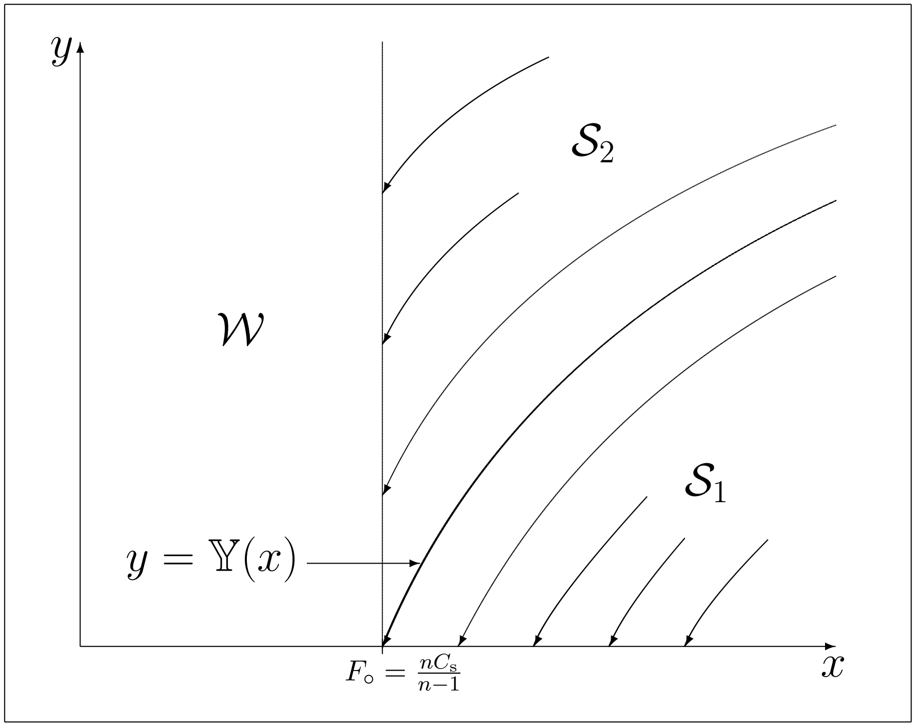
for some function $A : \mathbb{R}_+ \to \mathbb{R}$, where $n$ is the positive solution to the quadratic equation

$$\frac{1}{2}\sigma^2 \ell(\ell-1) + \mu\ell - \delta \equiv \frac{1}{2}\sigma^2\ell^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\ell - \delta = 0. \tag{41}$$

- To determine $A$ and $F$, we postulate that $w$ is $C^{2,1}$, in particular, along the free-boundary $F$, which yields

$$F(y) = \frac{nC_{\mathrm{s}}}{n-1} =: F_\circ, \tag{42}$$

$$A(y) = e^{-\lambda n y}\int_0^y e^{\lambda n u}\frac{1}{n}\left(\frac{n-1}{nC_{\mathrm{s}}}\right)^{n-1} du = \frac{1}{\lambda n^2}\left(\frac{n-1}{nC_{\mathrm{s}}}\right)^{n-1}(1 - e^{-\lambda n y}). \tag{43}$$




The regions providing the optimal strategy when $\overline{T} = \infty$. If the stock price takes values in the "waiting" region $\mathcal{W}$, then it is optimal to take no action. If the stock price at time 0 is inside the "selling" region $\mathcal{S}_1$, then it is optimal to sell all available shares immediately. If the stock price at time 0 is inside the "selling" region $\mathcal{S}_2$, then it is optimal to liquidate an amount that would cause the stock price to drop to $F_\circ$ and then keep on selling until all shares are exhausted by just preventing the stock price to rise above $F_\circ$.

- We conclude with the candidate for a solution to the HJB equations (36)–(37) given by

$$w(x,y) = \begin{cases} 0, & \text{if } y = 0 \text{ and } x > 0, \\ A(y)x^n, & \text{if } y > 0 \text{ and } x \leq F_\circ, \\ A\big(y - \mathbb{Y}(x)\big)F_\circ^n + \frac{x - F_\circ}{\lambda} - C_\mathrm{s}\mathbb{Y}(x), & \text{if } y > 0 \text{ and } F_\circ < x < F_\circ e^{\lambda y}, \\ \frac{1}{\lambda}x\big[1 - e^{-\lambda y}\big] - C_\mathrm{s} y, & \text{if } y > 0 \text{ and } F_\circ e^{\lambda y} \leq x, \end{cases} \tag{44}$$

where

$$\mathbb{Y}(x) = \frac{1}{\lambda} \ln \frac{x}{F_\circ}, \quad \text{for } x > 0. \tag{45}$$

- The value function identifies with this solution, namely,

$$v(x,y) = w(x,y) \quad \text{for all } (x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+. \tag{46}$$

- If we define

$$\xi_t^{\mathrm{s}\star} = y \wedge \sup_{0\le s\le t} \frac{1}{\lambda}\big[\ln x + B_s - \ln F_\circ\big]^+, \quad \text{for } t > 0, \tag{47}$$

where

$$B_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t, \tag{48}$$

then the following statements are true:

(I) If $\mu - \frac{1}{2}\sigma^2 \ge 0$, then $(\xi^{\mathrm{s}\star}, 0)$ is an optimal liquidation strategy.

(II) If $\mu - \frac{1}{2}\sigma^2 < 0$, then $(\xi^{\mathrm{s}\star}, 0)$ is not an admissible liquidation strategy. In this case, if we define

$$\xi_t^{\mathrm{s}\star j} = \xi_t^{\mathrm{s}\star}\mathbf{1}_{\{t\le n\}} + y\mathbf{1}_{\{n<t\}}, \quad \text{for } t > 0 \text{ and } j \ge 1, \tag{49}$$

then $(\xi^{\mathrm{s}\star j}, 0)$ gives rise to a sequence of $\varepsilon$-optimal strategies.