

The structured singular value problem arising in Robust Control: Complexity and Convex relaxation algorithms

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and Mathematical Sciences

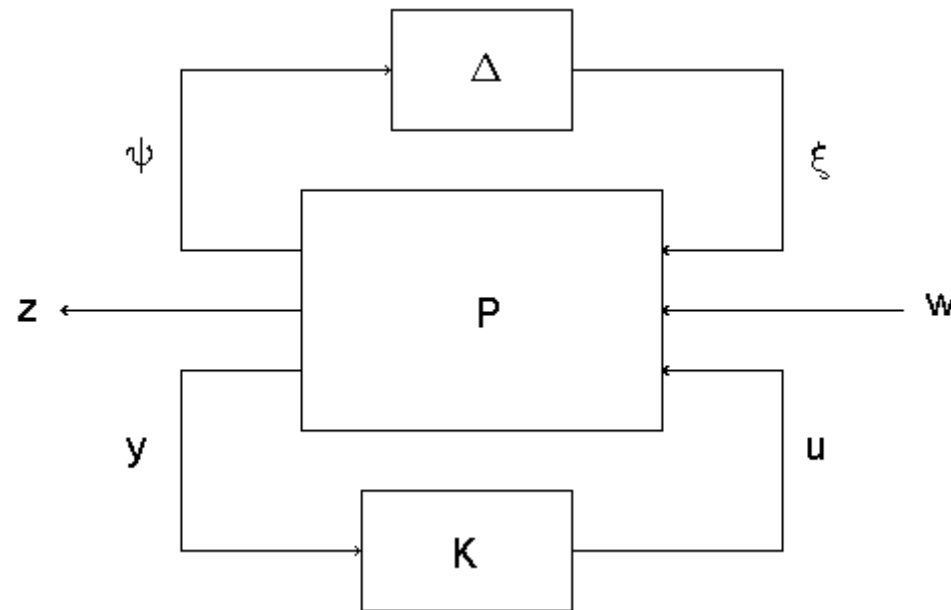
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Main Points

- The robust control problem
- Definition of structured singular value
- Examples and applications (outside control)
- Convex relaxation and upper bounds
- Reduction of duality gap
- A similar approach for an unconnected problem (QIP)
- Conclusions and further work
- (Robust Distributed Control)

The robust control problem: Generalized regulator I



Robust Control Problem:

$$\inf_{K \in \mathcal{S}} \sup_{\Delta \in \Delta} \|T_{zw}\|_{\infty}$$

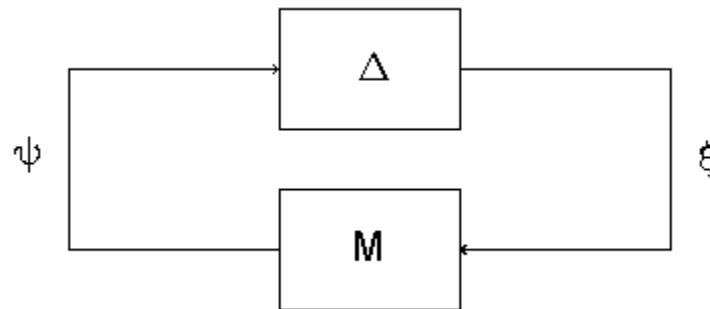
where Δ structured uncertainty set and $\|T_{zw}\|_{\infty} = \sup_{\omega \in \mathcal{R}} \sigma_1(T(j\omega))$.

The robust control problem: Generalized regulator II

- When (fixed) Δ absorbed in P : Standard \mathcal{H}_∞ control problem:

$$\inf_{K \in \mathcal{S}} \|T_{zw}(\Delta)\|_\infty$$

- When (fixed) K is absorbed in P :



Structured Singular value problem:

$$\sup\{\|\Delta\|_\infty : \Delta \in \Delta, \det(I - \Delta M) \neq 0 \quad \forall \omega \in \mathcal{R}\}$$

Structured Singular value: Definition

- Let $M \in \mathcal{C}^{n \times n}$, $\det(M) \neq 0$ (w.l.g.). Define $\Delta \subseteq \mathcal{C}^{n \times n}$:

$$\Delta = \{\text{diag}[\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}, \Delta_{s+1}, \dots, \Delta_{s+f}] :$$

$$\delta_i \in \mathcal{C}, \Delta_{s+j} \in \mathcal{C}^{m_j \times m_j}, 1 \leq i \leq s, 1 \leq j \leq f\}$$

where $\sum_{i=1}^s r_i + \sum_{j=1}^f m_j = n$.

- Define the “*structured distance to singularity*” of M :

$$r_{\Delta}(M) = \min\{\|\Delta\| : \Delta \in \Delta, \det(I - \Delta M) = 0\}$$

and the “*structured singular value*” of M :

$$\mu_{\Delta}(M) = r_{\Delta}^{-1}(M)$$

Structure Singular value: Applications

- *Robust control:*
 - (i) Optimal robust control problem
 - (ii) Structured distance to stability
 - (ii) distance to uncontrollability, etc.
- *Numerical Analysis*
 - (i) Structured condition number
 - (ii) approximate GCD of polynomials,
 - (iii) pseudo-spectra of operators, etc.
- *Robust optimization/estimation*

Structured singular value: Example (I)

Example: Let:

$$a(s) = s^m + a_{m-1}s^{m-1} + \dots + a_0, \quad b(s) = s^n + b_{n-1}s^{n-1} + \dots + b_0$$

be co-prime polynomials. Let

$$\hat{a}(s) = s^m + (a_{m-1} + \delta_{m-1})s^{m-1} + \dots + (a_0 + \delta_0)$$

$$\hat{b}(s) = s^n + (b_{n-1} + \epsilon_{n-1})s^{n-1} + \dots + (b_0 + \epsilon_0)$$

and define:

$$\gamma = \max\{ |\delta_0|, \dots, |\delta_{m-1}|, |\epsilon_0|, \dots, |\epsilon_{n-1}| \}$$

Find:

$$\gamma^* = \min\{ \gamma : (\hat{a}, \hat{b}) \text{ have a common root} \}$$

Structured singular value: Example (II)

Define Sylvester matrix:

$$R_{m,n}(a, b) = \begin{pmatrix} 1 & \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_0 & 0 & \cdots & 0 \\ 0 & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 \\ 1 & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_0 & 0 & \cdots & 0 \\ 0 & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix}$$

Then:

$$(a, b) \text{ co-prime} \Leftrightarrow \text{Rank}(R_{m,n}(a, b)) = m + n$$

Structured singular value: Example (III)

“Perturbed” Sylvester matrix:

$$R_{m,n}(a, b) + \begin{pmatrix} 0 & \delta_{m-1} & \delta_{m-2} & \cdots & \delta_0 & 0 & \cdots & 0 \\ 0 & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 \\ 0 & \epsilon_{n-1} & \epsilon_{n-2} & \cdots & \epsilon_0 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 \end{pmatrix}$$

Uncertainty has “structure” and calculation of γ^* is a ssv problem.

SSV: Convex relaxations and upper bounds (I)

- In general, the computation of μ_{Δ} is a non-convex NP-hard problem.
- To compute upper bound first note that:

$$\begin{aligned}\mu_{\Delta}^{-1}(M) &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - M\Delta) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(M^{-1} - \Delta) = 0\} \\ &\geq \min\{\|\Delta\| : \det(M^{-1} - \Delta) = 0\} \\ &= \|M\|^{-1}\end{aligned}$$

so that $\mu_{\Delta}(M) \leq \|M\|$.

- To tighten bound define:

$$\mathbf{D} = \{D = D^* > 0 : D\Delta = \Delta D \forall \Delta \in \Delta\}$$

Then for any $D \in \mathbf{D}$, $\Delta \in \Delta$: $D^{1/2}\Delta = \Delta D^{1/2}$.

SSV: Convex relaxations and upper bounds (II)

- Hence:

$$\begin{aligned}\mu_{\Delta}^{-1}(M) &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - M\Delta) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - MD^{-1/2}\Delta D^{1/2}) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - D^{1/2}MD^{-1/2}\Delta) = 0\} \\ &= \mu_{\Delta}^{-1}(D^{1/2}MD^{-1/2}) \geq \|D^{1/2}MD^{-1/2}\|^{-1}\end{aligned}$$

- We then have relaxation bound:

$$\mu_{\Delta}(M) \leq \min_{D \in \mathbf{D}} \|D^{1/2}MD^{-1/2}\| := \bar{\mu}_{\Delta}(M)$$

Since:

$$\|D^{1/2}MD^{-1/2}\| \leq \gamma \Leftrightarrow \gamma^2 D - M^*DM \geq 0$$

the computation of $\bar{\mu}_{\Delta}(M)$ now reduces to a convex programming problem.

SSV: Convex relaxations and upper bounds (III)

- Simplify problem by absorbing optimal D -matrices and scaling:

(i) Solve: $\mu_0 = \min_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\|$ and let D_o be a minimizer.

(ii) Re-define: $M \leftarrow \mu_0^{-1} D_o^{1/2} M D_o^{-1/2}$ so that: $\bar{\mu}_\Delta(M) = \|M\| = 1$.

(iii) Let $M \in \mathcal{C}^{n \times n}$ have (ordered) SVD:

$$M = U \Sigma V^* = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}$$

where $\Sigma_2 := \text{diag}(\sigma_{m+1}, \dots, \sigma_n)$, $\sigma_{m+1} < 1$.

SSV: Convex relaxations and upper bounds (IV)

- When is convex bound equal to $\mu(= 1)$?

Theorem: Let $M = U_1 V_1^* + U_2 \Sigma_2 V_2^*$ with

$$U_1 = \begin{pmatrix} A_1 \\ \vdots \\ A_s \\ E_1 \\ \vdots \\ E_f \end{pmatrix}, \quad V_1 = \begin{pmatrix} B_1 \\ \vdots \\ B_s \\ H_1 \\ \vdots \\ H_f \end{pmatrix}$$

compatibly partitioned with Δ . Let:

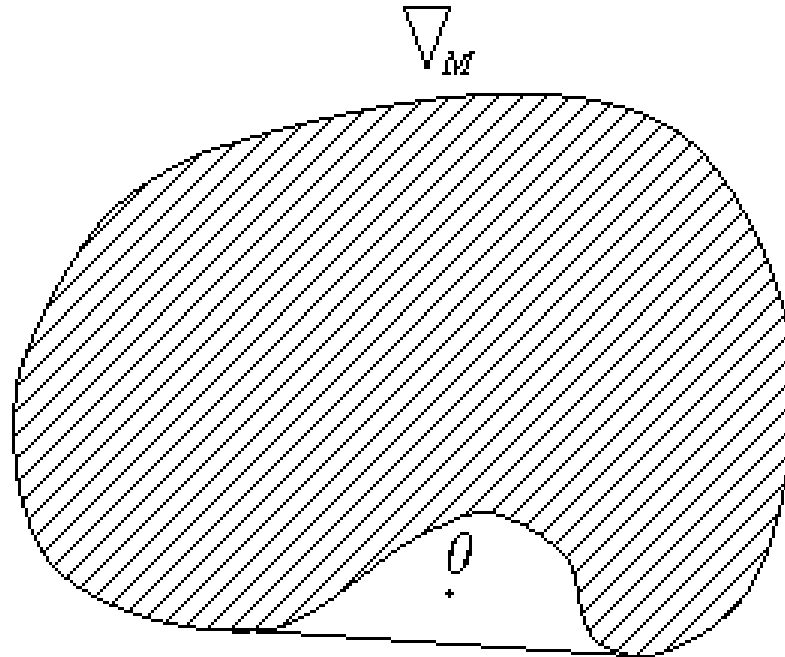
$$\begin{aligned} \nabla_M = \{ \text{diag}(P_1^\eta, \dots, P_s^\eta, p_{s+1}^\eta I_{m_1}, \dots, p_{s+f-1}^\eta I_{m_{f-1}}, 0_{m_r}) \\ : \eta \in \mathbf{C}^m, \|\eta\| = 1 \} \end{aligned}$$

where

$$P_i^\eta = A_i \eta \eta^* A_i^* - B_i \eta \eta^* B_i^*, \quad p_{s+j}^\eta = \eta^* (E_j^* E_j - H_j^* H_j) \eta$$

Then:

$$\mu_{\Delta}(M) = \bar{\mu}_{\Delta}(M) \Leftrightarrow 0 \in \text{co}(\nabla_M)$$



Fact: (i) $2s + f \leq 3 \Rightarrow 0 \in \text{co}(\nabla_M)$. (ii) Multiplicity of largest s.v. of M is $m = 1 \Rightarrow 0 \in \text{co}(\nabla_M)$. Therefore assume that $m > 1$.

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (I)

- *Main idea:* Embed uncertainty within larger structured uncertainty set for which distance to singularity can be calculated more easily.
- Consider perturbation matrix $\Delta \in \mathcal{C}^{n \times n}$:

$$\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}, \quad \Delta_{11} \in \mathcal{C}^{m \times m}$$

- Solve, or at least calculate upper bounds:

$$\phi_1 := \max\{\rho(V_1^* \Delta U_1) : \Delta \in \mathcal{B}\Delta\} \leq \bar{\phi}_1 \leq 1$$

$$\phi_2 := \max\{\|V_1^* \Delta U_1\| : \Delta \in \mathcal{B}\Delta\} \leq \bar{\phi}_2 \leq 1$$

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (II)

Then:

$$\begin{aligned} r_\Delta &= \mu_\Delta^{-1}(M) \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - \Delta M) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - \Delta U \Sigma V^*) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - V^* \Delta U \Sigma) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(\Sigma^{-1} - V^* \Delta U) = 0\} \\ &\geq \min\{\|\Delta\| : \rho(\Delta_{11}) \leq \bar{\phi}_1 \|\Delta\|, \|\Delta_{11}\| \leq \bar{\phi}_2 \|\Delta\|, \det(\Sigma^{-1} - \Delta) = 0\} \\ &:= \hat{r}_\Delta(M) := \hat{\mu}_\Delta^{-1}(M) \end{aligned}$$

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (III)

- $\hat{\mu}_\Delta(M)$ can be obtained analytically and $\hat{\mu}_\Delta(M) \leq \bar{\mu}_\Delta(M)$.
- Calculation of ϕ_1 ($\bar{\phi}_1$): Since

$$\begin{aligned}\phi_1 &= \max_{\Delta \in \mathcal{B}\Delta} \rho(V_1^* \Delta U_1) = \max_{\Delta \in \mathcal{B}\Delta} \rho(\Delta U_1 V_1^*) \\ &:= \max_{\Delta \in \mathcal{B}\Delta} \rho(\Delta M_0) \\ &= \mu_\Delta(M_0)\end{aligned}$$

this is a ssv problem *of reduced rank*.

- Calculation of ϕ_2 ($\bar{\phi}_2$): Use convex relaxation (LMI):

$$\phi_2^2 = \max_{\Delta \in \mathcal{B}\Delta} \|V_1^* \Delta U_1\|^2 \leq \min_{D \in \mathbf{D}, D - V_1 V_1^* \geq 0, \gamma I - U_1^* D U_1 \geq 0} \gamma = \bar{\phi}_2^2$$

- Lemma: $\phi_2 = \bar{\phi}_2$; $\inf_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\| = 1 \Rightarrow \phi_2 = \bar{\phi}_2 = 1$.

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (IV)

The problem thus reduces to:

$$\hat{\mu}_\Delta^{-1} = \min\{\|\Delta\| : \det(\Sigma^{-1} - \Delta) = 0, \rho(\Delta_{11}) \leq \bar{\phi}_1 \|\Delta\|, \|\Delta_{11}\| \leq \bar{\phi}_2 \|\Delta\|\}$$

Theorem: Let $0 \leq \bar{\phi}_1 \leq \bar{\phi}_2 \leq 1$. Then

$$\hat{\mu}_\Delta^{-1} = \min\{\gamma > 1 : \|f(\gamma, \Psi)\| = \sigma_{m+1}^{-1}\}$$

where: $f(\gamma, \Psi) = (\gamma I - \Psi)(\gamma^{-1}I - \Psi)^{-1}$ and

$$\Psi_{ij} = \begin{cases} 0, & j < i \\ \phi_1, & j = i \\ -\left(\frac{\phi_1}{\phi_2}\right)^{j-i-1} \frac{\phi_2^2 - \phi_1^2}{\phi_2}, & j > i \end{cases}$$

Also: (i) $\bar{\phi}_1 < 1 \Rightarrow \mu_\Delta(M) < 1$ and $\hat{\mu}_\Delta(M) < 1$ (ii) $\mu_\Delta(M) = \hat{\mu}_\Delta(M)$ iff $\exists \Delta \in \Delta$ s.t.

$$V_1^* \Delta U_1 = \hat{\mu}_\Delta(M) W^* \Psi W,$$

for unitary W . □

Note: Ψ is Toeplitz upper triangular. For example, if $m = 4$,

$$\Psi = \begin{bmatrix} \phi_1 & \alpha & \beta & \gamma \\ 0 & \phi_1 & \alpha & \beta \\ 0 & 0 & \phi_1 & \alpha \\ 0 & 0 & 0 & \phi_1 \end{bmatrix}$$

with

$$\alpha = \frac{\phi_2^2 - \phi_1^2}{\phi_2}, \beta = -\frac{\phi_1 \phi_2^2 - \phi_1^2}{\phi_2 \phi_2}, \gamma = \left(\frac{\phi_1}{\phi_2}\right)^2 \frac{\phi_2^2 - \phi_1^2}{\phi_2}$$

Note: The calculation of $\hat{\mu}_\Delta(M)$ is a $4m \times 4m$ e-value problem. Set $a = a_{m+1}$, and $\Psi_s = \Psi + \Psi^*$. Then $a = \|(\gamma I - \Psi)(\gamma^{-1}I - \Psi)^{-1}\| \Rightarrow$

$$\det \left(\gamma I - \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ a^2 I - a^2 \Psi_s & (a^2 - 1) \Psi^* \Psi & \Psi_s & \end{bmatrix} \right) = 0$$

Hence $\hat{\mu}_\Delta^{-1}$ is the smallest real e-value larger than one of a $4m \times 4m$ matrix.

Quadratic Integer Programming problem (QIP)

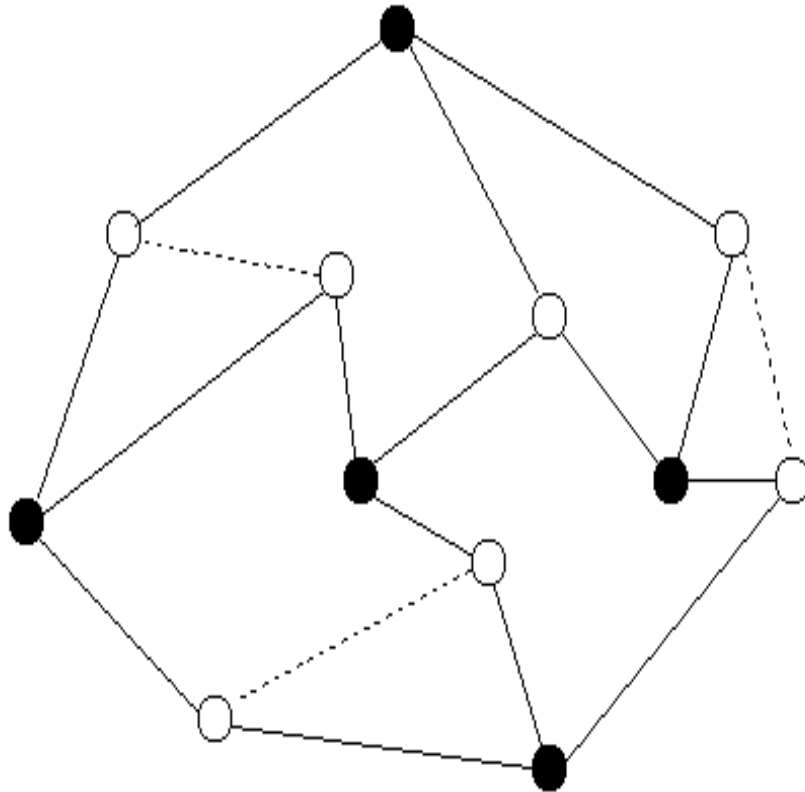
- QIP Problem has the form:

$$\gamma := \max\{x'Qx : x \in \{-1, 1\}^n\}$$

where $Q = Q'$ of mixed inertia in general.

- This needs 2^n (or 2^{n-1}) function evaluations which grows exponentially in n .
- Classical NP-hard problem with many applications in Control, Statistics, Graph Theory (“max-cut”), etc.

Example: Max-cut problem (I)



Given a graph defined by:

- Vertex set $V = \{1, 2, \dots, n\}$
- Edge set $E = \{(i, j) : i, j \in V, i \neq j\}$.
- For $S \subseteq V$ capacity is number of edges connecting a node in S to a node not in S .
- Max-Cut problem: Find $S \subseteq V$ with maximum capacity

Example: Max-cut problem (II)

- Define adjacency matrix:

$$Q_{ij} = 1 \quad \text{if } (i, j) \in E$$
$$= 0 \quad \text{otherwise}$$

- Specify cut by vector $x \in \mathcal{R}^n$:

$$x_i = 1 \quad \text{if } i \in S$$
$$= -1 \quad \text{otherwise}$$

- If $(i, j) \in S \times \bar{S}$, $1 - x_i x_j = 2$. Hence capacity:

$$c(x) = \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n (1 - x_i x_j) Q_{ij}$$

- Thus max-cut can be formulated as: $\min\{x'Qx : x \in \{-1, 1\}^n\}$.

QIP: Convex relaxation (I)

- Let $D = \text{diag}(D)$, $D - Q \geq 0$. Then for all $x \in \{-1, 1\}^n$:

$$\begin{aligned}x'Qx &= \underbrace{x'Dx - \text{trace}(D)}_{\text{trace}(D)} - x'(D - Q)x + \text{trace}(D) \\ &= -x'(D - Q)x + \text{trace}(D) \quad (x_i^2 = 1) \\ &\leq \text{trace}(D) \quad (D - Q \geq 0)\end{aligned}$$

- This results in the convex relaxation:

$$\begin{aligned}\gamma &= \max\{x'Qx : x \in \{-1, 1\}^n, Q = Q'\} \\ &\leq \min\{\text{trace}(D) : D = \text{diag}(D), D - Q \geq 0\} \\ &:= \bar{\gamma}\end{aligned}$$

- Let D_o be the (unique) optimal solution and define V such that $R(V) = \text{Ker}(D_o - Q)$, $V^*V = I_r$. If $r = 1$ duality gap is zero.

QIP: Convex relaxation (II)

- Solve RRQIP problem:

$$\gamma_r = \frac{1}{n} \max\{x'VV'x : x \in \{-1, 1\}^n\}$$

Then: $\gamma = \bar{\gamma} \Leftrightarrow \gamma_r = 1$.

- Provided $\gamma_r < 1$,

$$\gamma < \bar{\gamma} - n(1 - \gamma_r)\lambda_+(D_o - Q) < \bar{\gamma}$$

where

$$\lambda_+(D_o - Q) := \min_{\{i: \lambda_i(D_o - Q) > 0\}} \lambda_i(D_o - Q)$$

- Note that $\text{Rank}(VV') = r < n$ (typically $r \ll n$).
- Reduced-rank problem has complexity $O(n^r)$ and can be solved efficiently using theory of zonotopes, hyperplane arrangements and linear programming [Avis, Fukuda].

Common characteristics of two approaches (I)

- μ -problem:

$$\mu^{-1} = \min_{\Delta \in \Delta, \det(I - \Delta M) = 0} \|\Delta\|$$

is NP-hard.

- Formulate and solve dual:
Get (μ_0, D_0) s.t.

$$\min_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\|$$

$$M \leftarrow \mu_0^{-1} D_0^{1/2} M D_0^{-1/2}$$

- QIP problem:

$$\gamma := \max_{x \in \{-1, 1\}^n} x' Q x$$

is NP-hard

- Formulate and solve dual:
Get $(\bar{\gamma}, D_0)$ s.t.

$$\bar{\gamma} = \min_{D = \text{diag}(D), D - Q \geq 0} \text{trace}(D)$$

$$\text{s.t. } D = \text{diag}(D), D - Q \geq 0$$

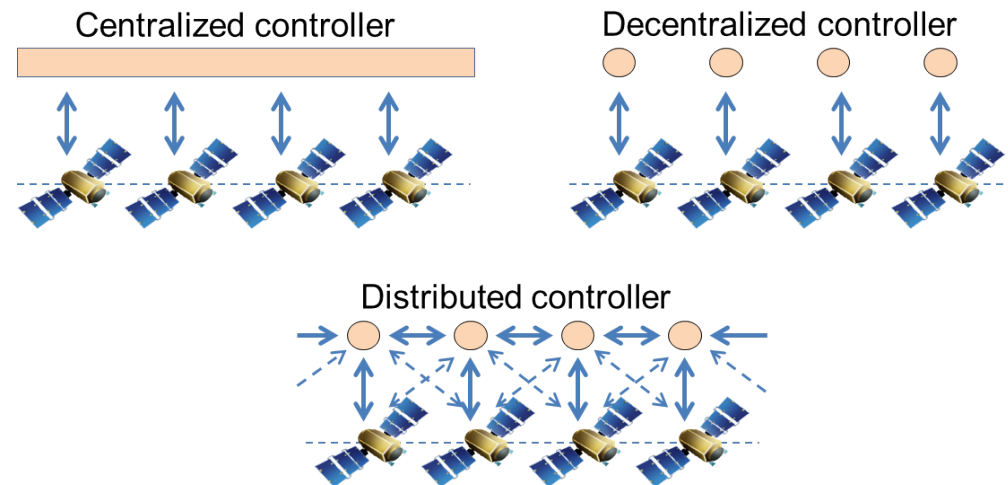
Common characteristics of two approaches (II)

- Sufficient condition for zero duality gap: $\mu = \bar{\mu}$ if $m := \text{null}(I - MM^*) = 1$.
- Duality gap can be breached if we can solve a rank- m μ problem (or get a bound less than 1):
$$\max_{\Delta \in \mathcal{B}\Delta} \rho(\Delta U_1 V_1^*)$$
- Improved bound obtained via the solution of an eigenvalue problem of dimension $4m \times 4m$.
- Sufficient condition for zero duality gap: $\gamma = \bar{\gamma}$ if $r := \text{null}(D_o - Q) = 1$.
- Duality gap can be breached if we can solve a rank- r QIP problem of the form
$$\gamma_r = \frac{1}{n} \max_{x \in \{-1, 1\}^n} x' V V' x.$$
- Improved bound can be obtained by calculating the eigenvalues of a symmetric matrix.

Conclusions and Open Issues

- Development of efficient algorithms for low-rank μ problems.
- Exploit similarities in solutions. Can method be generalized to a large class of problems involving convex relaxations (e.g. new class of primal/dual algorithms)?
- Can approach be combined with probabilistic relaxation methods?

Distributed Control over networks (with Ms I. Tomic)



- Objective: Develop robust distributed control schemes over networks.
- Interconnection of agents defined by graph.
- Each agent can utilise information only from neighbouring nodes. This can be corrupted by noise and be subjected to time-delays.

- Graph is deterministic but possibly time-varying (to model disruption in communications)
-

Some fundamental questions:

- How can centralized control schemes be extended in distributed environment?
- Can (robust) stability margins of centralized scheme be exploited to enforce stability of distributed solution?
- What is the optimum interconnection topology? Can the corresponding deterioration in "cost" be quantified?
- What are the effects of partial information (output vs state)? How can robust observers/Kalman filters be designed without excessive deterioration of stability margins?