School of Social Sciences
CITY UNIVERSITY LONDON

Department of Economics

The Two Sector Endogenous Growth Model: An Atlas

Michael Ben-Gad[1]
City University London

Department of Economics
Discussion Paper Series
No. 09/02

[1] Economics Department, City University, London EC1V OHB Phone: +44 (0) 20 7040 4582 e-mail: *mbengad@city.ac.uk*

The Two Sector Endogenous Growth Model: An Atlas

Michael Ben-Gad
Department of Economics
City University
Northampton Square
London EC1V 0HB, UK*

January 6, 2009

Abstract

In this paper we investigate the underlying structure of the Lucas (1988) endogenous growth model. We derive analytically, the restrictions on the parameter space that are necessary and sufficient for the existence of balanced growth paths and saddle-path stable local dynamics. We demonstrate that in contrast to the original model, with the addition of an external effect and depreciation in the human capital sector, the Lucas model can be made consistent with the high degrees of intertemporal elasticities of substitution increasingly estimated in the empirical literature—even if there is a significant degree of increasing returns to scale in the physical production sector of the economy. Finally we demonstrate that for a given baseline rate of steady state growth, with the inclusion of modest degrees of depreciation and external effects to the human capital production process, the model can accommodate the widest possible range of economies—including those characterized by low discount factors, high elasticities of intertemporal substitution, increasing returns in the final goods sector, and also both the high rates of population growth and steady state per-capita output growth we observe in many parts of the world today.

JEL classification: O41; D62
Keywords: Two-Sector Endogenous Growth Model; Intertemporal Elasticity of Substitution; Necessary and Sufficient Conditions

*I wish to thank Garret Jones, José Olmo and Saqib Jafarey as well as participants of the 2008 meetings of the Society for Economic Dynamic at MIT for helpful suggestions and advice. e-mail: *mbengad@city.ac.uk*

1 Introduction

Since publication of Lucas (1988), the two-sector endogenous growth model with human capital has featured in increasing numbers of applications in macroeconomics. This paper has two aims. First, to provide an atlas of sorts for the Lucas model—one extended to include sector-specific external effects and depreciation in both sectors—by mapping out analytically, the precise restrictions on the parameter space necessary and sufficient for the existence of balanced growth paths and for the existence of saddle-path stable equilibria in their vicinity. Second, the Lucas model in its original form has trouble accommodating values for the intertemporal elasticity of substitution that are significantly higher than one, a problem that becomes more acute for high rates of population growth or high rates of per-capita output growth. With the growing body of empirical evidence pointing towards higher values for the intertemporal elasticity of substitution in mind, this paper demonstrates that by adding a degree of sector-specific external effects and/or depreciation to the human capital sector, the two-sector model can be made consistent with high rates of intertemporal elasticity of substitution, as the well as high rates of population growth and high rates of output growth we commonly observe in many parts of the world.

In Section 2, we present the two-sector endogenous growth model with both depreciation and sector-specific external effects in each sector of the economy, and derive the laws of motion that characterize the model's dynamic behavior. Caballé and Santos (1993) analyze the two-sector model with depreciation in both sectors but only establish some general conditions for the existence of balanced growth paths. Xie (1994) includes external effects in the production sector, while abstracting from depreciation and external effects in the production of human capital. Unlike Caballé and Santos, Xie presents explicit bounds on the parameter space necessary and sufficient to guarantee balanced growth, however only by setting the intertemporal elasticity of substitution strictly equal to the reciprocal of the share of physical capital in the production sector. In this paper we do not impose this restriction.

In Section 3, we derive the steady state values for capital, consumption, and hours of market work. Following Benhabib and Perli (1994) and Ben-Gad (2003), we then use these values to analytically define the restrictions on the parameter space in terms of bounds on the subjective discount rate necessary and sufficient to ensure the existence of interior solutions to the representative agent's optimization program which support unique balanced growth paths. In Section 4, we further restrict the parameter space, by ruling out balanced growth paths characterized by unstable local dynamics.

Section 5 demonstrates the implications of our analytical results using numerical examples that focus on the behavior of the model in the usually problematic region where the intertemporal

elasticity of substitution is greater than one. Varying the magnitude of both external effects and the intertemporal elasticity of substitution, while fixing the other parameters of the model, we demonstrate that with the inclusion of external effects and depreciation the two-sector model is able to accommodate the high values for the intertemporal elasticity of substitution as estimated by Hansen and Singleton (1982), Amano and Wirjanto (1997), Mulligan (2002) and Gruber (2006) for the United States, or Hamori (1996) and Fuse (2004) for Japan.

Finally in Section 6, we restrict our attention to those portions of the parameter space most likely correspond to empirically relevant rates of growth. Fixing the baseline rate of steady state growth, we demonstrate that including modest degrees of depreciation and external effects to the human capital production process, enables us to calibrate the model for the widest possible range of economies—including those characterized by low discount factors, high elasticities of intertemporal substitution, increasing returns in the final goods sector, and high rates of population growth and steady state per-capita output growth.

2 The Model

The economy is composed of a large number of households whose behavior can be represented by the intertemporal maximization of an infinite-lived representative agent. This agent maximizes utility over time t, by choosing the dynamic path of consumption, c, and $u \in (0,1)$, the fraction of time as well as human capital h devoted to working in the final goods sector:

$$\max_{c} \int_0^\infty e^{(n-\rho)t} \frac{\sigma}{\sigma - 1} c^{1-1/\sigma} dt, \tag{P.1}$$

subject to the constraints:

$$\dot{k} = wuh + (r - \delta) k - c,$$

$$\dot{h} = \nu \left[(1-u) h\right]^{1-\gamma} \left[(1-\bar{u}) \bar{h}\right]^{\gamma} - \varepsilon h,$$

where σ is the constant rate of intertemporal elasticity of substitution, ρ a positive discount rate, n the natural rate of population growth, δ the rate of depreciation of physical capital k, r its rate of return, ε the rate of depreciation of human capital and w the wage rate. The terms $\bar{u} \in (0,1)$ and $\bar{h}$ are the time t share of time devoted to market work and the time t stock of human capital, aggregated over all the agents in the economy and expressed in per-capita terms—hence the term $\left[(1-\bar{u}) \bar{h}\right]^{\gamma}$ captures the efficiency enhancing external effects of that portion of the human capital stock employed in that sector, and the parameter γ regulates its magnitude. Time not devoted to work for wages is spent accumulating human capital—ν is the maximum rate at which human capital can be accumulated.

Physical goods are produced by a combination of physical capital and effective labor $\phi = uh$:

$$y = \left(\bar{u}\bar{h}\right)^{\beta} F\left(k, \phi\right), \tag{1}$$

where the term $\left(\bar{u}\bar{h}\right)^{\beta}$ captures the efficiency enhancing external effects of that portion of the human capital stock employed in the final goods sector. We assume that the function $F : R^2 \rightarrow R$ takes the constant returns, Cobb-Douglas form $F\left(k, \phi\right) = k^{\alpha}\phi^{1-\alpha}$. Internal factor returns are:

$$r = \left(\bar{u}\bar{h}\right)^{\beta} F_k\left(k, \phi\right), \tag{2}$$

$$w = \left(\bar{u}\bar{h}\right)^{\beta} F_{\phi}\left(k, \phi\right). \tag{3}$$

Unlike Lucas' aggregate external effects, we limit the scope of external effects to be sector-specific. Only the portion of human capital that is employed in a sector generates spill-over effects for that sector, but these are sufficient to generate both differential rates of steady state growth for the two types capital, and higher rental rates for human capital in rich countries. The most obvious spill-overs are likely to be the result of complementarities between the skills of workers—personnel in a sector interact and learn from each other. Efficiency of the final goods sector is certainly enhanced by increases in the total stock of knowledge—however, this may be knowledge produced by both domestic and foreign human capital sectors. Restricting spill-overs to be sector-specific obviates the need to distinguish between endogenous domestically produced human capital, and the foreign portion of human capital which is accumulating exogenously.[1]

The present value Hamiltonian that corresponds to the consumer's optimization problem is:

$$\begin{aligned} H\left(c, u, k, h, \lambda, \mu\right) &= e^{(n-\rho)t}\frac{\sigma}{\sigma-1}c^{1-1/\sigma} + \lambda\left[wuh + rk - c - \delta k\right]P.1 \\ &\quad + \mu\left[\left[\left(1-u\right)h\right]^{1-\gamma}\left[\left(1-\bar{u}\right)\bar{h}\right]^{\gamma} - \varepsilon h\right], \end{aligned} \tag{4}$$

where λ and μ are the costate variables for physical and human capital.

Inserting the values from (2) and (3), in place of r and w, the first order necessary conditions for an interior solution to the individual constrained optimization are:

$$\frac{e^{(n-\rho)t}}{c^{1/\sigma}} = \lambda, \tag{5}$$

$$\left(1-\alpha\right)\lambda h\left(\bar{u}\bar{h}\right)^{\beta} k^{\alpha}\left(uh\right)^{-\alpha} = \mu\left(1-\gamma\right)\nu\left(1-u\right)^{-\gamma} h^{1-\gamma}\left[\left(1-\bar{u}\right)\bar{h}\right]^{\gamma}, \tag{6}$$

[1] Paul and Siegel (1999) find strong evidence of sizeable increasing returns—two-thirds to almost three-quarters can be ascribed to agglomeration effects—sector specific external effects at the two-digit industry level. Harrison (1998) finds evidence of increasing returns but rejects spillovers between sectors and Benhabib and Jovanovic (1991), demonstrate that the source of any aggregate increasing returns to scale are not associated with the capital input. Finally, Durlauf *et. al.* (2008) finds strong evidence for the existence of production externalites in explaining cross-country differences in per-capita growth.

$$\alpha\left(\bar{u}\bar{h}\right)^{\beta}k^{\alpha-1}\left(uh\right)^{1-\alpha}\lambda-\delta=-\dot{\lambda}, \tag{7}$$

$$\mu\left[\left(1-\gamma\right)\nu\left(1-u\right)^{1-\gamma}h^{-\gamma}\left[\left(1-\bar{u}\right)\bar{h}\right]^{\gamma}-\varepsilon\right]+\lambda\left(1-\alpha\right)\left(\bar{u}\bar{h}\right)^{\beta}k^{\alpha}\left(uh\right)^{-\alpha}u=-\dot{\mu}, \tag{8}$$

plus the two transversality conditions,

$$\lim_{t\to\infty}\lambda k=0, \tag{9}$$

$$\lim_{t\to\infty}\mu h=0, \tag{10}$$

and the constraint that u falls within the unit interval. We define the parameter space Θ: $\theta\equiv\left(\alpha,\beta,\gamma,\delta,\varepsilon,\nu,\rho,\sigma,n\right)$, and $\theta\in\Theta$, where $\Theta=R^{2}_{++}\times R^{4}_{+}\times\left[0,1\right)^{3}$.

Setting $\bar{u}=u$ and $\bar{h}=h$, differentiating (5) with respect to time, and substituting into (7), the law of motion for per-capita consumption is:

$$\frac{\dot{c}}{c}=\sigma\left(\alpha k^{\alpha-1}\phi^{1-\alpha+\beta}-\rho-\delta\right). \tag{11}$$

The law of motion for per-capita physical capital is:

$$\frac{\dot{k}}{k}=k^{\alpha-1}\phi^{1-\alpha+\beta}-\frac{c}{k}-\delta-n, \tag{12}$$

Substituting the wage equation into (6) and differentiating with respect to t:

$$\frac{\dot{\mu}}{\lambda}-\frac{\mu}{\lambda^{2}}\dot{\lambda}=\frac{\alpha\left(1-\alpha\right)k^{\alpha-1}\phi^{\beta-\alpha}}{\left(1-\gamma\right)\nu}\dot{k}+\frac{\left(\beta-\alpha\right)\left(1-\alpha\right)k^{\alpha}\phi^{\beta-\alpha-1}}{\left(1-\gamma\right)\nu}\dot{\phi}. \tag{13}$$

Substituting (8) and (12) for $\dot{\mu}$ and $\dot{k}$ into (13) yields the law for motion of effective labor:

$$\frac{\dot{\phi}}{\phi}=\frac{\alpha}{\alpha-\beta}\left(\frac{\left(1-\gamma\right)\nu-\varepsilon}{\alpha}+\frac{1-\alpha}{\alpha}\left(n+\delta\right)-\frac{c}{k}\right). \tag{14}$$

The evolution of the economy is described by the system (11), (12) and (14) in the non-stationary variables c, k and ϕ. To make this system stationary, we define stationary consumption and physical capital: $\widetilde{c}=c\phi^{-\frac{1-\alpha+\beta}{1-\alpha}},\widetilde{k}=k\phi^{-\frac{1-\alpha+\beta}{1-\alpha}}$. The dynamic system reduces to two stationary laws of motion:

$$\frac{\dot{\widetilde{c}}}{\widetilde{c}}=\sigma\left(\alpha\widetilde{k}^{\alpha-1}-\delta-\rho\right)-\vartheta\left(\frac{\left(1-\gamma\right)\nu-\varepsilon}{\alpha}+\frac{1-\alpha}{\alpha}\left(n+\delta\right)-\frac{\widetilde{c}}{\widetilde{k}}\right), \tag{15}$$

and

$$\frac{\dot{\widetilde{k}}}{\widetilde{k}}=\widetilde{k}^{\alpha-1}+\left(\vartheta-1\right)\frac{\widetilde{c}}{\widetilde{k}}-\vartheta\frac{\left(1-\gamma\right)\nu-\varepsilon}{\alpha}-\frac{n+\delta}{\alpha-\beta}, \tag{16}$$

where $\vartheta=\frac{1-\alpha+\beta}{\alpha-\beta}\frac{\alpha}{1-\alpha}$.

3 Balanced Growth

The balanced growth paths of the economy are the solutions to the equations (15) and (16) when $\dot{\widetilde{c}}=\dot{\widetilde{k}}=0$. Differentiating $\phi = uh$ with respect to time: $\dot{\phi} = \dot{u}h + u\dot{h}$, setting $\dot{u} = 0$, and combining the law of motion for human capital in (4) with (14), (15), and (16) yields the steady state fraction of hours devoted to production in the final goods sector:

$$u^* = \frac{\rho - n - (\eta - \gamma)\,\nu + \eta\varepsilon}{(1-\eta)\,\nu}, \tag{17}$$

where $\eta = \frac{(1-\alpha+\beta)(\sigma-1)}{(1-\alpha)\sigma}$ is the product of the curvature of the utility function, and the ratio of the social marginal product of human capital to the private marginal product of human capital. The steady state growth rate of physical output, consumption wages and physical capital is:

$$\kappa = \frac{(1-\alpha+\beta)\,((1-\gamma)\,\nu - \varepsilon - \rho + n)}{(1-\alpha)\,(1-\eta)}, \tag{18}$$

and the steady state growth rate of human capital is $\frac{(1-\gamma)\nu-\varepsilon-\rho+n}{1-\eta}$.

Setting the left hand sides of (15) and (16) equal to zero we solve for balanced growth consumption and capital:

$$\widetilde{c}^* = \left[\frac{(1-\alpha)(n+\delta)}{\alpha} + \frac{(1-\alpha+\beta-\eta)((1-\gamma)\nu-\varepsilon) + (\alpha-\beta)(\rho\ -n)}{\alpha(1-\eta)}\right]\widetilde{k}^*, \tag{19}$$

and

$$\widetilde{k}^* = \left[\frac{\alpha\,(1-\alpha)\,(1-\eta)}{(1-\alpha+\beta-(1-\alpha)\,\eta)\,(n+\delta+(1-\gamma)\,\nu-\varepsilon) - \beta\,(\delta+\rho)}\right]^{\frac{1}{1-\alpha}}. \tag{20}$$

To ensure the existence of interior solutions along the balanced growth path, the representative agent cannot be so impatient that he allocates all available time to immediate production, or so patient that all participation in the labor market is postponed indefinitely as the maximum accumulation of human capital is pursued. Therefore, as in Benhabib and Perli (1994) and Ben-Gad (2003), we use bounds on the discount rate to describe the restrictions on preferences necessary to ensure that the fraction of hours worked is on the unit interval and that the steady state rate of growth is positive.

We define the two disjoint subspaces of the parameter space $\Theta_1, \Theta_2 \subset \Theta$:

$$\Theta_1 \equiv \{\theta \in \Theta |\ n + (\eta-\gamma)\,\nu - \eta\varepsilon < \rho < n + (1-\gamma)\,\nu - \varepsilon \text{ and } \eta < 1\}, \tag{21}$$

$$\Theta_2 \equiv \{\theta \in \Theta |\ n + (1-\gamma)\,\nu - \varepsilon < \rho < n + (\eta-\gamma)\,\nu - \eta\varepsilon \text{ and } \eta > 1\}. \tag{22}$$

Proposition 1 *If $\theta \in \{\Theta_1, \Theta_2\}$, the steady state growth rate $\kappa > 0$, the steady state fraction of hours worked $u^* \in (0,1)$, and the steady state stock of physical capital $\widetilde{k}^* > 0$.*

Proof: If $\eta < 1$ then from (18) $\kappa > 0$ iff $\rho < n + (1-\gamma)\nu - \varepsilon$ which implies $\rho < n + (1-\gamma)\nu - \eta\varepsilon$ and $u^* < 1$ from (17). Furthermore if $\eta < 1$ and $\rho < n + (1-\gamma)\nu - \varepsilon$ then $\rho < (n + (1-\gamma)\nu - \varepsilon)\left(1 + \frac{(1-\alpha)(1-\eta)}{\beta}\right) + \frac{(1-\alpha)(1-\eta)\delta}{\beta}$ which from (20) implies $\widetilde{k}^* > 0$. If $\eta > 1$ then from (18) $\kappa > 0$ iff $\rho > n + (1-\gamma)\nu - \varepsilon$ which implies $\rho > n + (1-\gamma)\nu - \eta\varepsilon$ and $u^* < 1$ from (17). Furthermore if $\eta > 1$ and $\rho > n + (1-\gamma)\nu - \varepsilon$ then $\rho > (n + (1-\gamma)\nu - \varepsilon)\left(1 + \frac{(1-\alpha)(1-\eta)}{\beta}\right) + \frac{(1-\alpha)(1-\eta)\delta}{\beta}$ which from (20) implies $\widetilde{k}^* > 0$. Finally, from (17), $u^* > 0$ iff $\eta < 1$ and $\rho < n + (\eta-\gamma)\nu - \eta\varepsilon$, or $\eta > 1$ and $\rho > n + (\eta-\gamma)\nu - \eta\varepsilon$. ■

The conditions in Proposition 1 are necessary but not sufficient to ensure the existence of an interior balanced growth path. For example if $\alpha = 0.6$ $\beta = 0.31$, $\gamma = 0.28$, $\delta = 0.03$, $\varepsilon = 0.15$, $\nu = 0.2$, $\rho = 0.03$, $\sigma = 4$, and $n = 0.02$, then $\widetilde{k}^*$=147.355, u^*=0.0085, κ=0.0857, however consumption is negative: $\widetilde{c}$ =-0.00185. Therefore to ensure that steady state consumption $\widetilde{c}^* > 0$ we define the subsets $\Theta_1^A, \Theta_1^B, \Theta_1^C \subset \Theta_1$ and $\Theta_2^A, \Theta_2^B, \Theta_2^C \subset \Theta_2$:

$$\Theta_1^A \equiv \{\theta \in \Theta_1 | n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\nu - \varepsilon) - \zeta < \rho < n + (1-\gamma)\nu - \varepsilon \text{ and } \alpha > \beta\}, \quad (23)$$

$$\Theta_1^B \equiv \{\theta \in \Theta_1 |\ n + (\eta-\gamma)\nu - \eta\varepsilon < \rho < n + (1-\gamma)\nu - \varepsilon \text{ and } \alpha < \beta\}, \quad (24)$$

$$\Theta_1^C \equiv \{\theta \in \Theta_1 | n + (\eta-\gamma)\nu - \eta\varepsilon < \rho < n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\nu - \varepsilon) - \zeta \text{ and } \alpha < \beta\}, \quad (25)$$

$$\Theta_2^A \equiv \{\theta \in \Theta_2 | n + (1-\gamma)\nu - \varepsilon < \rho < n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\nu - \varepsilon) - \zeta \text{ and } \alpha > \beta\}, \quad (26)$$

$$\Theta_2^B \equiv \{\theta \in \Theta_2 | n + (1-\gamma)\nu - \varepsilon < \rho < n + (\eta-\gamma)\nu - \eta\varepsilon \text{ and } \alpha < \beta\}, \quad (27)$$

$$\Theta_2^C \equiv \{\theta \in \Theta_2 | n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\nu - \varepsilon) - \zeta < \rho < n + (\eta-\gamma)\nu - \eta\varepsilon \text{ and } \alpha < \beta\}. \quad (28)$$

where $\zeta = \frac{1-\eta}{\alpha-\beta}(1-\alpha)(n+\delta)$. Note that in addition to $\Theta_1 \cap \Theta_2 = \varnothing$, also $\Theta_i^A \cap \Theta_i^j = \varnothing$, where $i \in \{1,2\}$ and $j \in \{B, C\}$.

Proposition 2 *If $\alpha \neq \beta$, the necessary and sufficient condition for the existence of an interior balanced growth path is: $\theta \in \Theta_1^A \cup \left(\Theta_1^B \cap \Theta_1^C\right) \cup \Theta_2^A \cup \left(\Theta_2^B \cap \Theta_2^C\right)$.*

Proof: From (19), for a positive valued $\widetilde{k}^*$, then $\widetilde{c}^* > 0$ iff $\rho\frac{\alpha-\beta}{1-\eta} > \frac{\alpha-\beta}{1-\eta}n - \left(1 - \frac{\alpha-\beta}{1-\eta}\right)((1-\gamma)\nu - \varepsilon) - (1-\alpha)(n+\delta)$. If $\frac{\alpha-\beta}{1-\eta} > 0$ then $\widetilde{c}^* > 0$ iff $\theta \in \Theta_1 \cup \Theta_2$ and $\rho > n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\nu - \varepsilon) - \frac{1-\eta}{\alpha-\beta}(1-\alpha)(n+\delta)$, hence $\theta \in \Theta_1^A \cup \left(\Theta_2^B \cap \Theta_2^C\right)$. If $\frac{\alpha-\beta}{1-\eta} < 0$, then $\widetilde{c}^* > 0$ iff $\theta \in \Theta_1 \cup \Theta_2$ and $\rho < n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\nu - \varepsilon) - \frac{1-\eta}{\alpha-\beta}(1-\alpha)(n+\delta)$, hence $\theta \in \left(\Theta_1^B \cap \Theta_1^C\right) \cup \Theta_2^A$. ■

The sets Θ_1 and Θ_2 are separated in Θ by a hyperplane defined by the set Θ_3:

$$\Theta_3 \equiv \{\theta \in \Theta | \rho = n + (1-\gamma)\nu - \varepsilon \text{ and } \eta = 1\}. \quad (29)$$

If $\theta \in \Theta_3$ the numerators and denominators in both (19) and (20) are both equal to zero, implying the existence of an infinite number of balanced growth paths. For the case of $\alpha = \beta$, see Corollary 1 in Appendix 8.1.

4 Dynamics and Equilibria

The results in the previous section demonstrate the conditions for balanced growth paths to be both interior and unique. However, the equilibrium paths that converge to these growth paths are only unique if the dynamic system has a saddle path structure. To find the local stability properties of the reduced system in the neighborhood of the balanced growth paths, we linearize the system (15) and (16). The Jacobian of the linearized system evaluated at the balanced growth path is given by:

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & -\mathbf{J}_{11} + \frac{\beta(\rho+\delta)\sigma-(1-\alpha+\beta)((1-\gamma)\nu+n+\delta-\varepsilon)}{1-\eta} \\ \frac{\beta}{\alpha(1-\alpha+\beta)}\mathbf{J}_{11} & -\frac{\beta}{1-\alpha+\beta}\mathbf{J}_{11} - \frac{(1-\alpha)(n+\delta)+(1-\alpha+\beta)((1-\gamma)\nu-\varepsilon)}{\alpha-\beta} \end{bmatrix}, \tag{30}$$

where $\mathbf{J}_{11} = \vartheta\frac{\widetilde{c}^*}{\widetilde{k}^*}$, and the value of $\frac{\widetilde{c}^*}{\widetilde{k}^*}$ is defined from (19) (see Appendix).[2] If the Jacobian of the reduced system has eigenvalues of opposite signs, we conclude that at least in the neighborhood of the balanced growth path, all competitive equilibria are determinate (locally unique). If both eigenvalues are negative, all paths converge to the balanced growth path and any point in its vicinity qualifies as a competitive equilibrium, and if both eigenvalues are positive, all paths diverge from the balanced growth path and violate the transversality conditions.

Proposition 3 *In the neighborhood of a balanced growth path competitive equilibrium are unique iff* $\theta \in \Theta_1^A \cup \left(\Theta_2^B \cap \Theta_2^C\right)$.

Proof: The determinant of $\mathbf{J}$ is:

$$|\mathbf{J}| = \frac{\alpha\,(1-\alpha)\,(1-\eta)\,\sigma}{\alpha-\beta}\widetilde{c}^*\widetilde{k}^{*\alpha-1}. \tag{31}$$

which is negative if and only if $\eta < 1$ and $\alpha > \beta$, or $\eta > 1$ and $\alpha < \beta$. From (23)—(28) the determinant (31) is negative iff $\theta \in \Theta_1^A \cup \left(\Theta_2^B \cap \Theta_2^C\right)$ and the eigenvalues of $\mathbf{J}$ have opposite signs and equilibria are locally unique. ■

The implication of Proposition 3 is that the portions of the parameter space defined by Θ_1^B,Θ_1^C, and Θ_2^A, might support the existence of a unique balanced growth path, but the equilibria in the neighborhood of these balanced growth paths are either unstable or indeterminate. We can rule out the latter.

[2]The Jacobian $\mathbf{J}$ is not defined for $\alpha = \beta$. Henceforth we ignore this case.

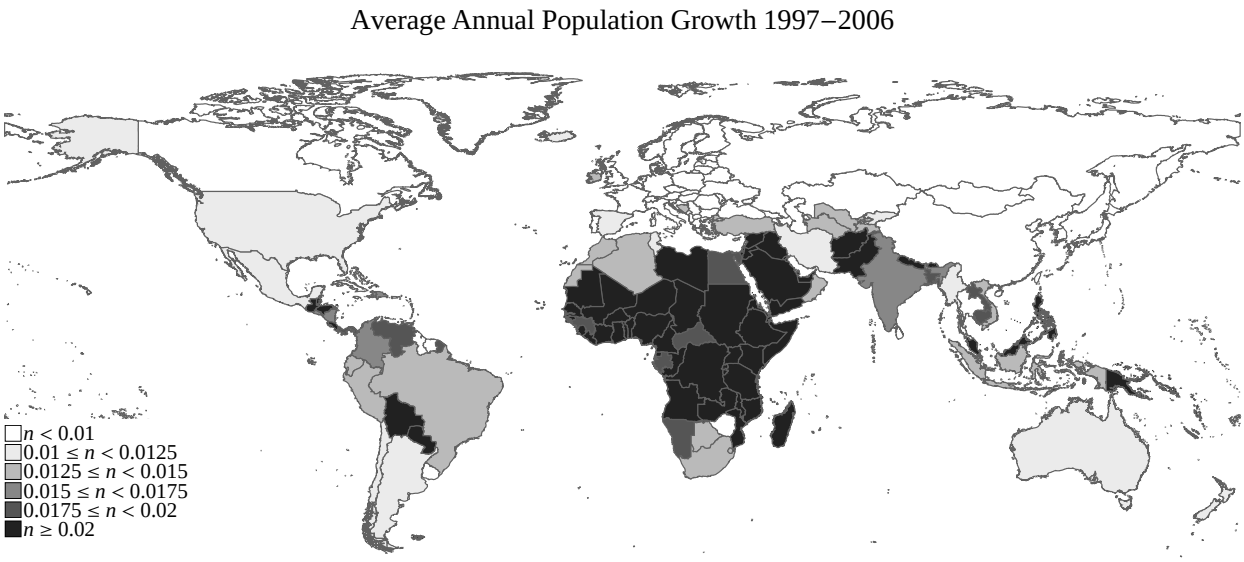

Figure 1: Population growth rates n, averaged over the decade between 1997-2006. *Source: Mathematica Research*

Proposition 4 *If* $\theta \in \Theta_2^A \cup \left(\Theta_1^B \cap \Theta_1^C\right)$*, all equilibria in the neighborhood of a balanced growth path are unstable.*

Proof: The trace of $\mathbf{J}$ is:

$$tr\mathbf{J} = \frac{\rho - n - \left(\left(1-\gamma\right)\nu - \varepsilon\right)\eta}{1-\eta}. \tag{32}$$

which is positive iff $\theta \in \Theta_1 \cup \Theta_2$ and negative otherwise. From (23)—(28) the determinant (31) is positive iff $\theta \in \Theta_2^A \cup \left(\Theta_1^B \cap \Theta_1^C\right)$. If the determinant and trace of $\mathbf{J}$ are positive, the eigenvalues of $\mathbf{J}$ are positive as well, and we can rule out multiple equilibria (indeterminacy).

5 Intertemporal Elasticities of Substitution Greater than One

The vast majority of models in the macroeconomic literature employ preferences characterized by constant intertemporal elasticity of substitution. In the DSGE literature these elasticities are in turn calibrated with values of σ that typically range between one half and one, a consequence of the fact that for time additive utility functions, the intertemporal elasticity of substitution is the reciprocal of the Arrow Pratt measure of relative risk aversion, which is usually assumed to fall within the range between one and two. By contrast, in the endogenous fertility literature (see Barro and Becker (1988), (1989)) the intertemporal elasticities of substitution are generally greater than one. These values can be found in some recent empirical studies on the United States and Japan.

Gruber (2006) estimates the intertemporal elasticity of substitution for individuals in the United States to be two. Hamori (1996) estimates the elasticity for Japanese consumers to be between one

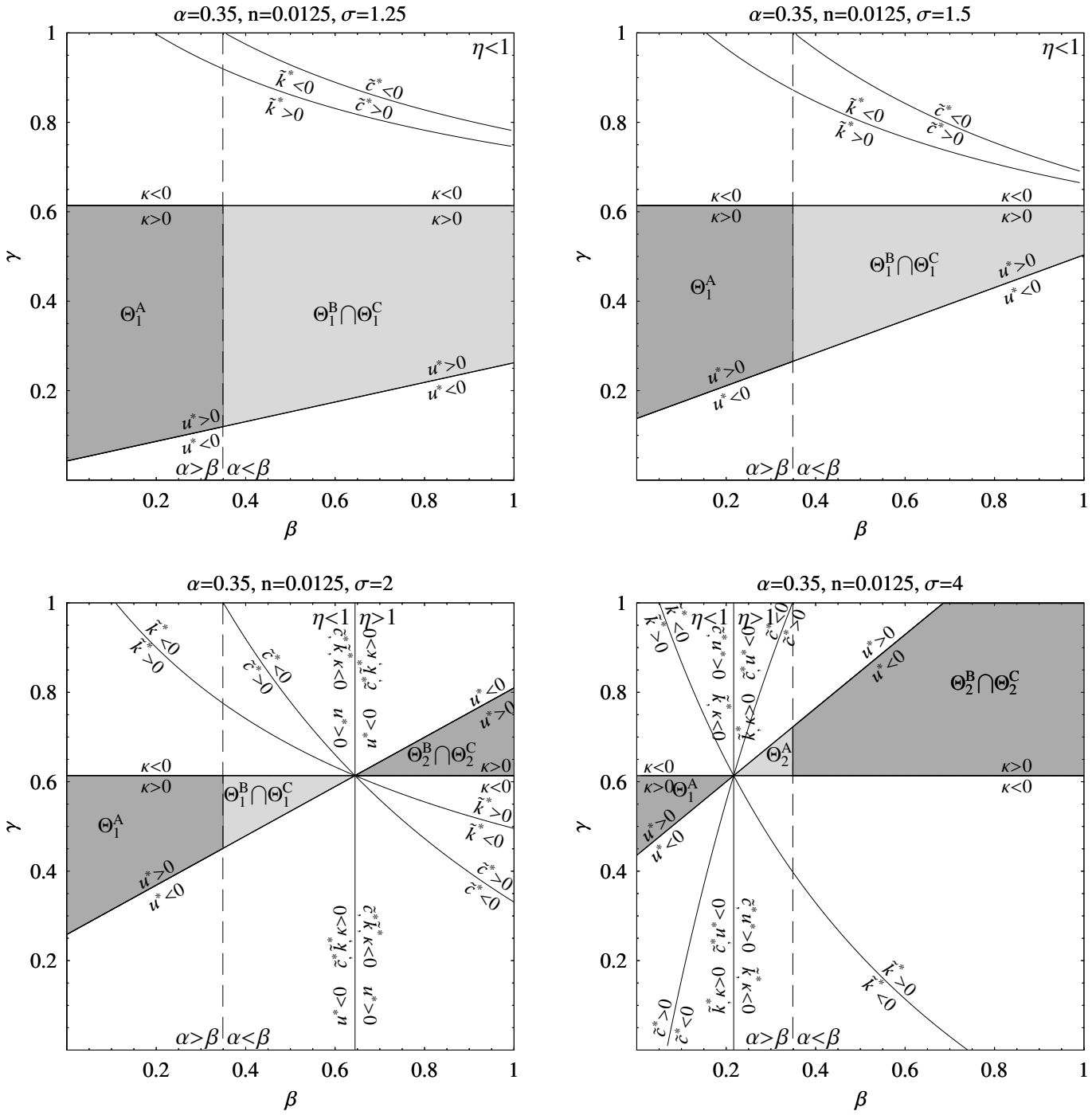

Figure 2: The parameter space for $\alpha = 0.35$ and $n = 0.0125$.

and two, and Fuse (2004) estimates the elasticity in Japan to be about four. Attanasio and Weber (1989), Mulligan (2002), Vissing-Jørgensen and Attanasio (2003), Bansal and Yaron (2004), Bansal, Kiku and Yaron (2007), and Hansen, *et. al.* (2007) all estimate high values for the intertemporal elasticity of substitution for the Epstein-Zin recursive utility function.[3] The question remains under what circumstances the two-sector endogenous growth model can cope with these higher elasticities.[4]

In Figure 2, we vary the magnitude of both external effects β and γ along the unit interval, for values of the intertemporal elasticity of substitution σ equal to 1.25, 1.5, 2, and 4, while holding the other parameters of the model fixed. We set the share of capital in output α equal to 0.35, the subjective discount rate ρ equal to 0.03, and the rates of depreciation for physical and human capital δ and ε equal to 0.065, and 0.05, respectively. We set the value of ν, the parameter that abstracting from depreciation, and in the absence of any activity devoted to production represents the maximum growth rate for human capital, equal to 0.175. Finally, we set the rate of population growth to n=0.0125, which approximates the recent experience of many middle-income countries such as Chile and Mexico at 0.0120, Indonesia and Peru at 0.0130, or Ecuador and South Africa at 0.0131 (see Figure 1).

In the upper left-hand panel of Figure 2 we consider the conditions for the existence of an interior balanced growth path, fixing the intertemporal elasticity of substitution to σ=1.25. The value of η is less than one, as long as $\beta < -\frac{1-\alpha}{1-\sigma}$, so throughout the portion of the parameter space under consideration $\eta < 1$. Both the areas shaded in dark and light gray, denoted Θ_1^A and $\Theta_1^B \cap \Theta_1^C$, respectively, represent the combinations of β and γ along the unit interval that satisfy all the necessary and sufficient conditions for interior balanced growth paths. However, from Proposition 3 only the former, shaded in dark gray, Θ_1^A, represents economies characterized by unique saddle-path stable equilibria. This area is bounded from below by the constraint $u^* > 0$, which begins where the value of γ equals 0.429 and rises linearly. Therefore, in order to ensure the existence of an interior balanced growth path, the value of the external effect in the human capital sector, γ, must be positive, but no greater than $1 - \frac{\rho-n+\varepsilon}{\nu}$—here equal to 0.614—otherwise the constraint that the growth rate is positive, $\kappa > 0$ is violated. Raising the value of β, necessitates

[3]Tversky and Kahneman (1992) estimate the coefficient of relative risk aversion to be 0.22, but in their non-expected utility framework, its reciprocal cannot be automatically treated as the intertemporal elasticity of substitution.

[4]Jones *et. al.* (2005) simulate the behavior of the endogenous growth model with elastic labor supply and fluctuations. Though they include depreciation in both sectors of the economy, the production function for human capital is linear at both the private and social levels—in their model the intertemporal elasticity of substitution cannot be much higher than 1.11. Nonetheless it only at this upper bound that the simulated standard deviation of the consumption-output ratio approaches that observed in US data.

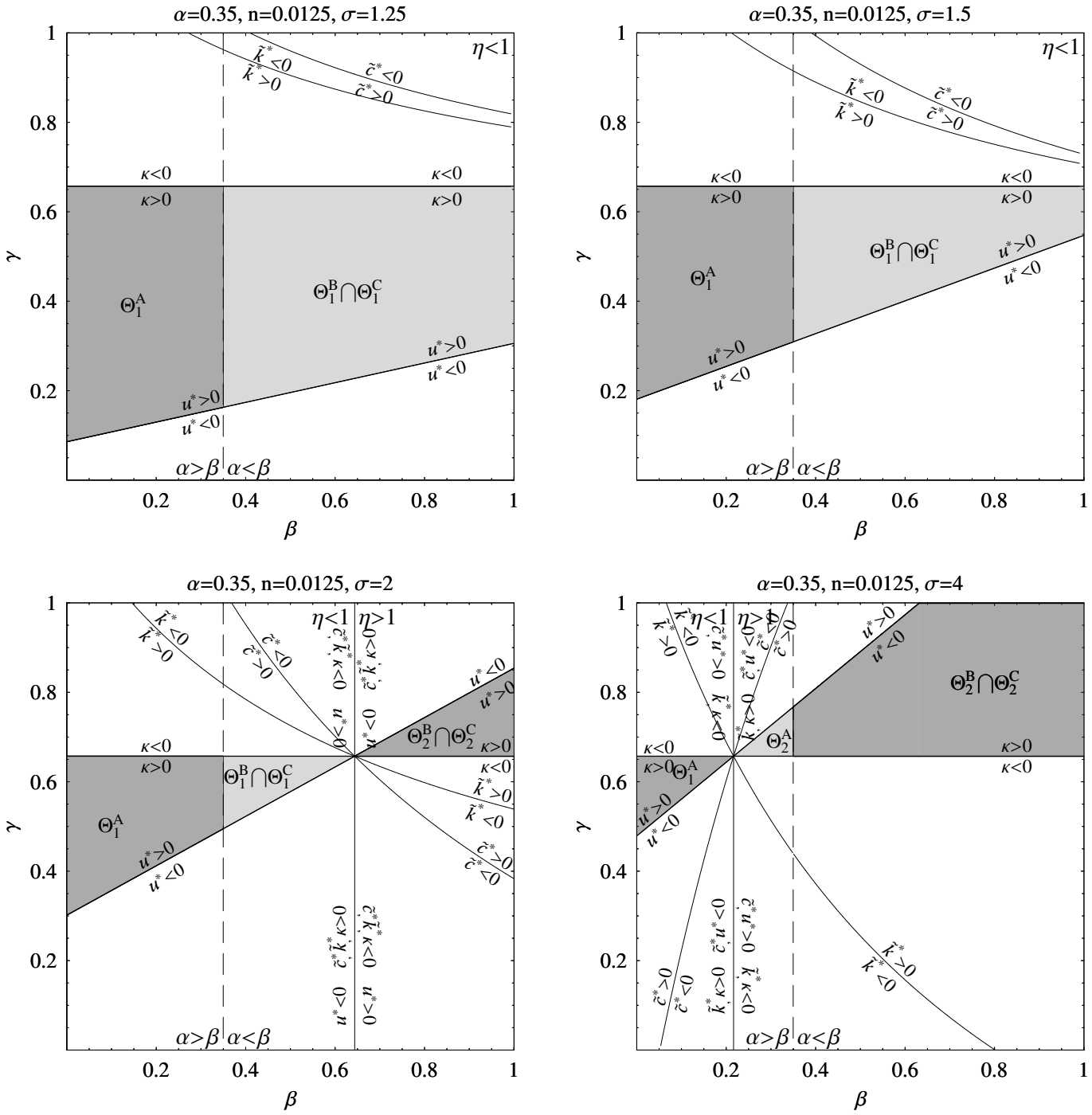

Figure 3: The parameter space for $\alpha = 0.35$ and $n = 0.02$.

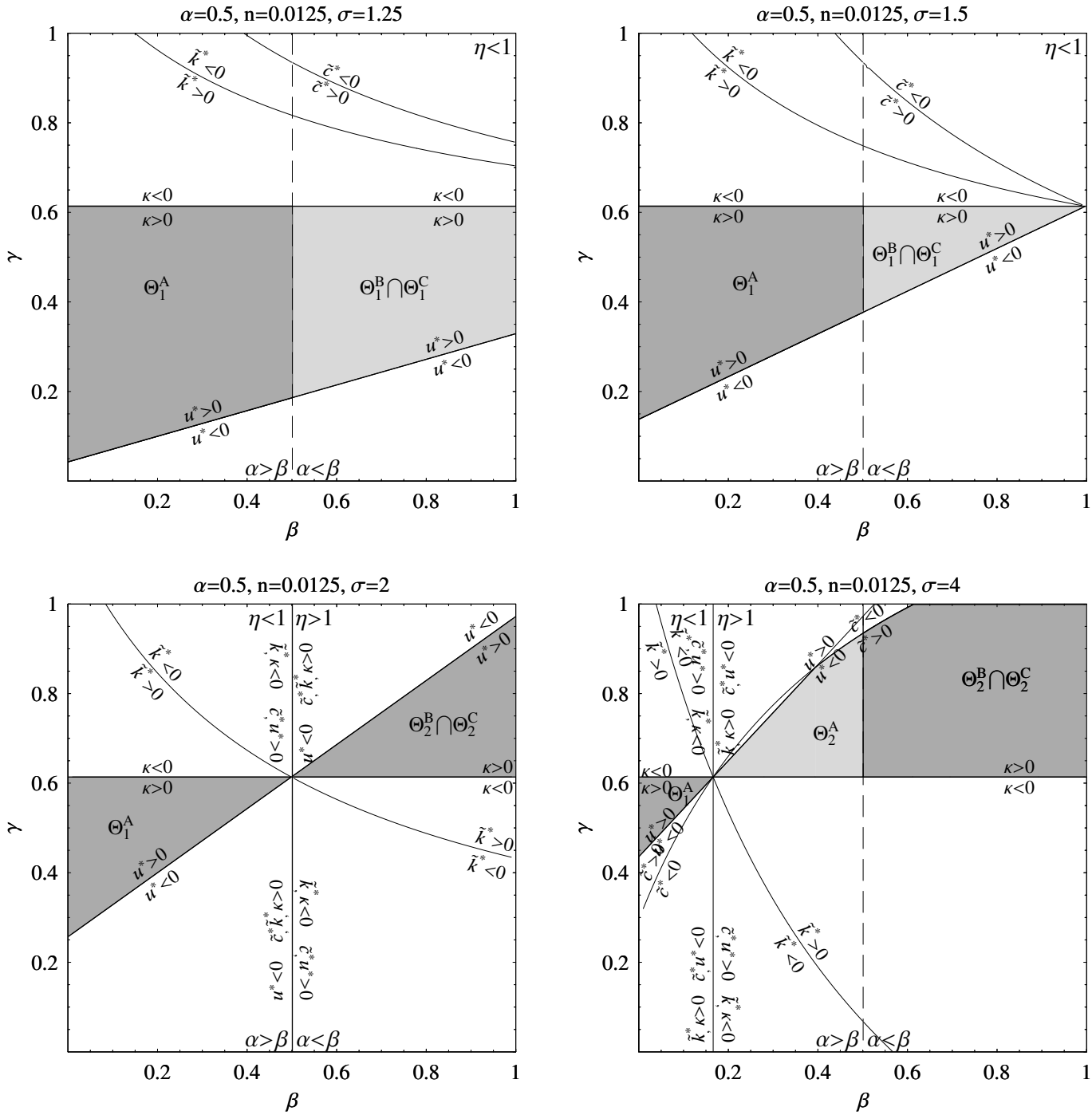

Figure 4: The parameter space for $\alpha = 0.5$ and $n = 0.0125$.

raising the value of γ as well. However since for $\sigma = 1.25$ and all values of $\beta < 1$, $\theta \in \Theta_1$, unless we are willing contemplate aggregate production functions homogenous of degree greater than two, no unique saddle path stable equilibria that converge to an interior balanced growth path emerge, unless the external effects β and γ, are no larger than 0.35 and 0.614, respectively.

Raising the value of σ to 1.5, narrows the range of values of β and γ that support the existence of interior balanced growth paths, and at minimum the value of γ, in the upper right-hand panel of Figure 2, must be at least 0.138. Raise the value of σ above 1.65 and components of Θ_2 appear for values of β <1. For example, in the lower left-hand panel of Figure 2, $\sigma = 2$. If $\beta = 0.65$, then η=1 and the set that corresponds to interior balanced growth paths reduces to a single point in the separating hyperplane Θ_3. Beyond this point, as the value of β grows beyond 0.65, the range of values for γ consistent with the existence of interior balanced growth paths, expands within the region defined by $\Theta_2^B \cap \Theta_2^C$. The regions that support saddle-path equilibria, Θ_1^A and $\Theta_2^B \cap \Theta_2^C$, are separated in Θ by closed neighborhoods.

Finally, raising the value of σ to 4, the upper range of recent estimates of the intertemporal elasticity of substitution cited above, greatly reduces the size of Θ_1^A. Furthermore, for all values of $\beta < \alpha$, η is less than one. Hence the region $\Theta_1^B \cap \Theta_1^C$ disappears, and the region Θ_2^A, where the balanced growth path is also unstable, emerges instead. The point on the separating hyperplane Θ_3, where η =1, is $\{\beta, \gamma\} = \{0.217, 0.614\}$.

A necessary and sufficient condition that ensures $u^* > 0$, is that $\sigma < \frac{(1-\alpha+\beta)(\nu-\varepsilon)}{(1-\alpha+\beta)(\nu-\varepsilon)-(1-\alpha)(\rho-n+\gamma\nu)}$. Given $\nu > \varepsilon$ and $\rho > n$, this condition is satisfied for all $\sigma < 1$, even if external effects are absent from both sectors. Raising the value of σ above one and beyond, the curvature of the human capital production function, regulated by the value of the parameter γ, becomes critical. Furthermore, the higher the rate of population growth, the higher the degree of curvature required as well. In the absence of external effects in either the human capital or the production sector, the aforementioned upper bound on σ reduces to $\frac{\nu-\varepsilon}{\nu-\varepsilon-\rho+n}$. Therefore in our example if n=0.01, the upper bound for σ is 1.190, and if n=0.0125 the upper bound drops to 1.163.

During the decade between 1997 and 2006 the annual rate of population growth in South America averaged 0.0143 per year, implying an upper bound of 1.144. The rate of population growth in South Asia averaged .0178, corresponding to an upper bound of 1.108; in the Middle East it averaged 0.0185, corresponding to an upper bound of 1.101; in Central America it averaged 0.02, corresponding to an upper bound of 1.087; and in Sub-Saharan Africa 0.258, corresponding to an upper bound of 1.035.[5] Furthermore, once we introduce increasing returns at the social level

[5] *South Asia:* Afghanistan, Bangladesh, Bhutan, India, Maldives, Nepal, Pakistan, Sri Lanka; *Middle East*: Algeria, Bahrain, Egypt, Iran, Iraq, Israel, Jordan, Kuwait, Lebanon, Libya, Mauritania, Morocco, Oman, Palestinian Territories, Qatar, Saudi Arabia, Sudan, Syria, Tunisia, Turkey, United Arab Emirates, Yemen; *Sub-Saharan Africa*:

generated by human capital external effects, all these upper bounds drop lower still.

Writing our necessary and sufficient conditions as bounds on the curvature of the human capital production function, an interior balanced growth path only exists for $\eta < 1$, if $\frac{(\nu-\varepsilon)\eta+n-\rho}{\nu} < \gamma < \frac{\nu-\varepsilon+n-\rho}{\nu}$; or for $\eta > 1$, when $\frac{\nu-\varepsilon+n-\rho}{\nu} < \gamma < \frac{(\nu-\varepsilon)\eta+n-\rho}{\nu}$. If $\beta = 0$ and $\sigma = 1.25$, the former bound corresponds to $0 < \gamma < 0.571$, if n=0.005; $0.029 < \gamma < 0.6$, if n=0.01; and $0.043 < \gamma < 0.614$, if n=0.0125. Similarly, the bounds that correspond to the rate of population growth in South America are $0.053 < \gamma < 0.625$; for South Asia, $0.073 < \gamma < 0.645$; for the Middle East, $0.077 < \gamma < 0.649$; for Central America, $0.086 < \gamma < 0.657$; and for Sub-Saharan Africa, $0.119 < \gamma < 0.690$. More generally, comparing the panels in Figure 2 where n=0.0125, with the panels in Figure 3 where n=0.02, the only difference is that all the admissible areas that correspond to interior balanced growth paths are shifted vertically by 0.0429. The higher the rate of population growth, the greater the degree of curvature in the human capital production required if the intertemporal elasticity of substitution is greater than one.

In Figure 4 we restore the rate of population growth to n=0.0125, but raise the share of capital in the production of physical output α, to 0.5. In the upper left-hand panel, σ=1.25 and if $\beta = 0$, the value of γ must once again fall between 0.043 and 0.614. Here the constraint that ensures $u^* > 0$ possesses a larger slope in the size of the external effect β, so the range of the parameter set consistent with the existence of interior balanced growth paths, is narrower than in Figure 2. If $\alpha = 0.5$, $\beta = 1$, and σ=1.5, then η=1, so Θ_3 is the very edge of the upper right-hand panel of Figure 4.

In the lower left-hand panel of Figure 4, we set σ=2 so that $\alpha = 1/\sigma$, and this corresponds to the version of the original Lucas model investigated in Xie (1994), though with the external effect from human capital restricted to be sector specific. Like Xie (1994) here too we do not encounter any unstable balanced growth paths, both $\Theta_1^B \cap \Theta_1^C$ and Θ_2^A disappear. However, because we rule out intersector spill-overs, so that only the human capital employed in the production sector generates positive external effects there, there is no region characterized by indeterminacy either, and all interior balanced growth paths are saddle path stable.

Finally, in the lower right-hand panel of Figure 4, for σ=4, the size of Θ_1^A reduces to a relatively small region, while the size of Θ_2^A, the range of parameter values that correspond to unstable dynamics expands when compared to its counterpart in Figure 2. Note also that in contrast to all

Angola, Benin, Botswana, Burkina Faso, Burundi, Cameroon, Cape Verde, Central Africam Republic, Chad, Comoros, Côte d'Ivoire, Democratic Republic of Congo, Djibouti, Equatorial Guinea, Eritrea, Ethiopia, Gabon, Gambia, Ghana, Guinea, Guinea Bissau, Kenya, Lesotho, Liberia, Madagascar, Malawi, Mali, Mauritania, Mauritius, Mayotte, Mozambique, Namibia, Niger, Nigeria, Republic of Congo, Reunion, Rwanda, Saint Helena, São Tomé Principe, Senegal, Seychelles, Sierra Leone, Somalia, South Africa, Swaziland, Tanzania, Togo, Uganda, Zambia, Zimbabwe.

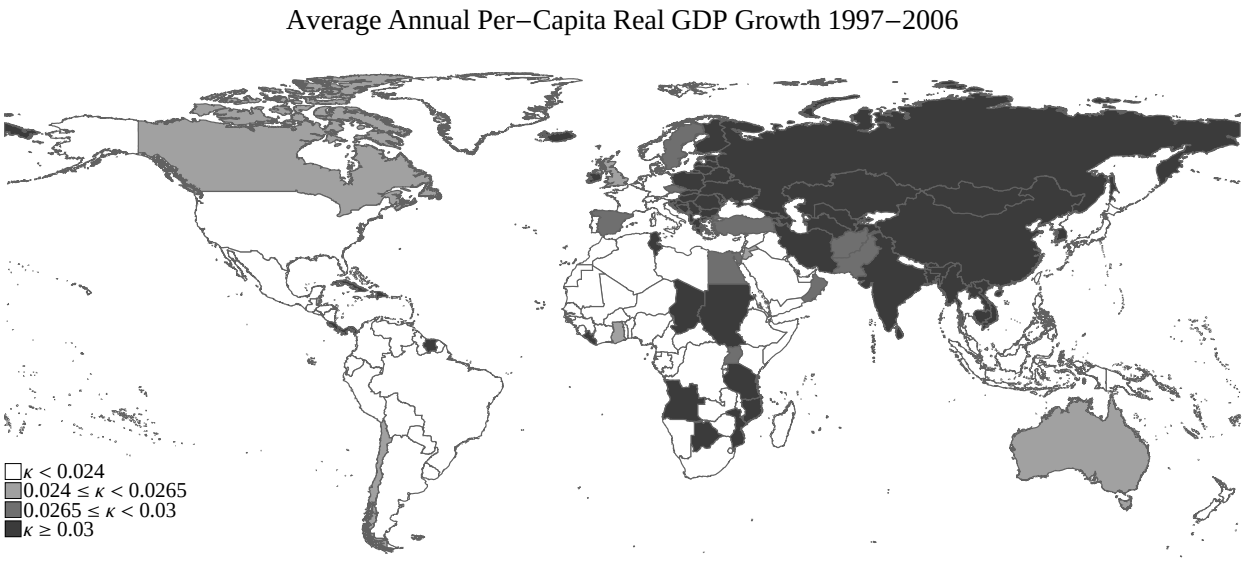

Figure 5: Per-capita real output growth κ, averaged over the decade between 1997-2006. *Source: Mathematica Research*

the other panels in Figures 2 and 4, the binding constraint is no longer either just $\kappa > 0$ or $u^* > 0$, but rather for high values of β and γ, $c^* > 0$ as well. This again demonstrates why Proposition 1, or merely restricting the parameter space to $\theta \in \{\Theta_1, \Theta_2\}$ is a necessary, but not at all a sufficient condition for the existence of an interior balanced growth path, and the further restrictions imposed in Proposition 2 are both necessary and sufficient.

6 Calibrating the Model for a Given Growth Rate

To better understand the nature of the parameter space and how it relates to empirically relevant rates of growth, we can solve (18) for one of the deep parameters of the model, then redefine the balanced growth path in terms of the steady state per-capita rate of growth κ. But which parameter should we replace? We are interested in analyzing the behavior of the model for different values of β, γ, ε, and σ, and the values of α, δ, ρ, and n are all parameters that can be easily calibrated using widely available data, as indeed can the growth rate κ. By contrast, there is very little direct evidence available that can be used to set the value of ν, the maximum possible growth rate for human capital at the social level, if every moment is devoted to its production (abstracting from its rate of depreciation). Therefore solving (18) for ν:

$$\nu = \frac{(1-\alpha)(1-\eta)\kappa + (1-\alpha+\beta)(\rho+\varepsilon-n)}{(1-\alpha+\beta)(1-\gamma)}, \tag{33}$$

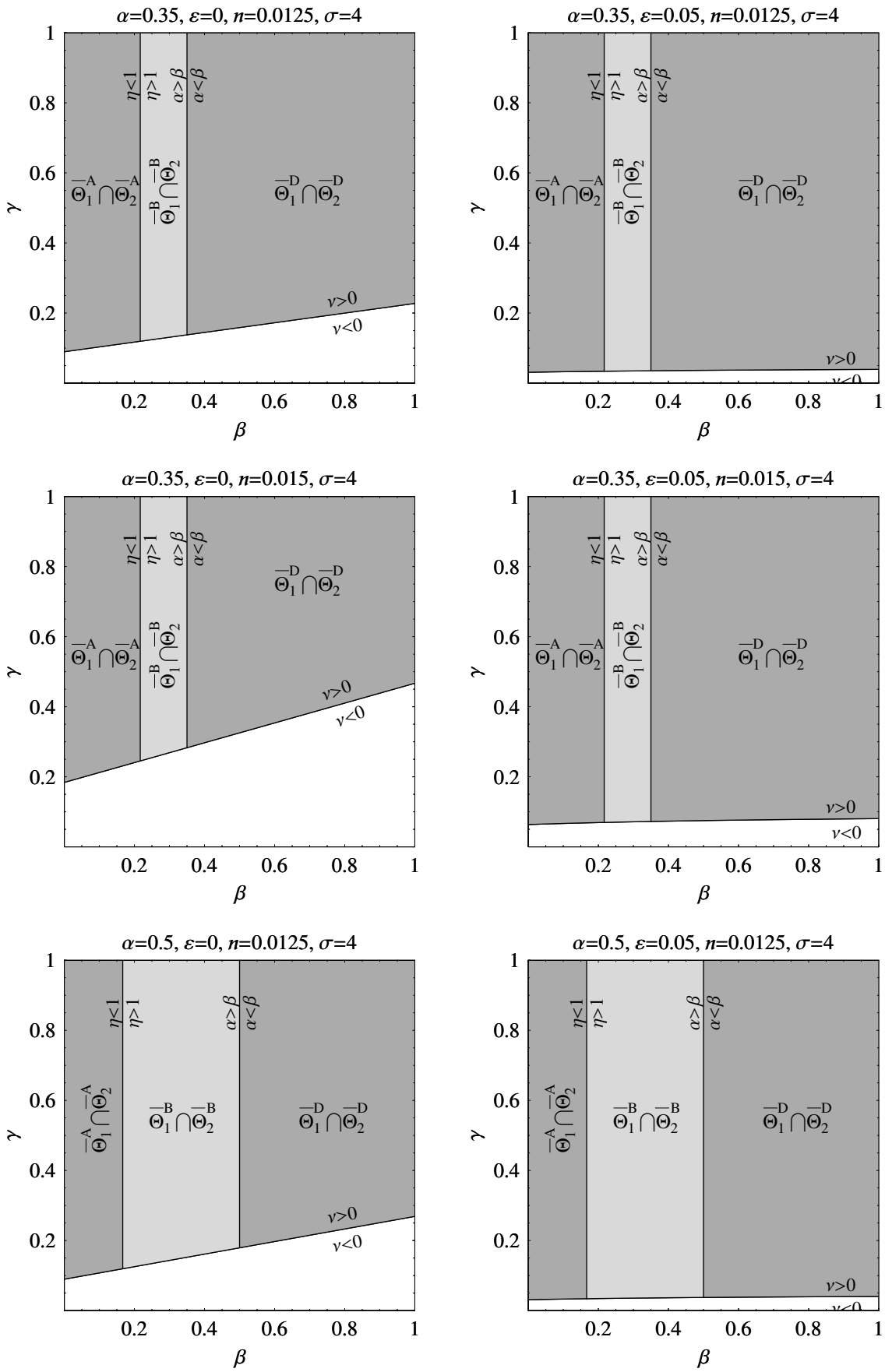

Figure 6: The parameter space for δ=0.065, κ=0.0265, ρ=0.03 and σ=4.

and then substituting (33) in (17) yields steady state hours worked:

$$u^* = \frac{(1-\alpha+\beta)(\kappa+(\rho-n+\gamma\varepsilon-\kappa)\sigma)+(1-\alpha)\gamma\kappa\sigma}{(1-\alpha+\beta)(1+(\rho-n+\varepsilon)\sigma)-\beta\kappa\sigma}, \tag{34}$$

in (19) yields steady state consumption:

$$\widetilde{c}^* = \left[\frac{1}{\alpha}(\delta+\rho+\frac{\kappa}{\sigma})-n-\delta-\kappa\right]\widetilde{k}^*, \tag{35}$$

and (20) yields steady state physical capital:

$$k^* = \left(\frac{\alpha\sigma}{\kappa+(\rho+\delta)\sigma}\right)^{\frac{1}{1-\alpha}}, \tag{36}$$

all in terms of the steady state growth rate κ.

Clearly from (36), if α,δ,κ,ρ, and σ are all positive, $k^* > 0$. We redefine the parameter space $\bar{\Theta}$: $\bar{\theta} \equiv (\alpha,\beta,\gamma,\delta,\varepsilon,\kappa,\rho,\sigma,n)$, and $\bar{\theta} \in \bar{\Theta}$, where $\bar{\Theta} = R^2_{++} \times R^4_{+} \times [0,1)^3$ and define the subsets $\bar{\Theta}_1, \bar{\Theta}_2, \bar{\Theta}_3 \subset \bar{\Theta}$:

$$\bar{\Theta}_1 \equiv \left\{\bar{\theta} \in \bar{\Theta} |\ \rho > n-\gamma\varepsilon-\frac{(1-\alpha)\gamma\kappa}{1-\alpha+\beta}-\frac{1-\sigma}{\sigma}\kappa\right\}, \tag{37}$$

$$\bar{\Theta}_2 \equiv \left\{\bar{\theta} \in \bar{\Theta} |\ \rho > \alpha(n+\delta+\kappa)-\delta-\frac{\kappa}{\sigma}\right\}, \tag{38}$$

$$\bar{\Theta}_3 \equiv \left\{\bar{\theta} \in \bar{\Theta} |\ \rho > n-\varepsilon+\frac{\beta\kappa}{1-\alpha+\beta}-\frac{\kappa}{\sigma}\right\}. \tag{39}$$

Proposition 5 *If $\alpha \neq \beta$, the necessary and sufficient condition for the existence of an interior balanced growth path is: $\bar{\theta} \in \bar{\Theta}_1 \cap \bar{\Theta}_2$.*

Proof: From (33), $\nu > 0$ iff $\bar{\theta} \in \bar{\Theta}_1$. From (34), $u^* > 0$ iff $\bar{\theta} \in \left(\bar{\Theta}\backslash\bar{\Theta}_1\right) \cup \bar{\Theta}_3$. However $\bar{\Theta}_3 \subset \bar{\Theta}_1$. Finally, from (35), $\widetilde{c}^* > 0$ iff $\bar{\theta} \in \bar{\Theta}_2$. ■

We further subdivide $\bar{\Theta}_1$ and $\bar{\Theta}_2$:

$$\bar{\Theta}_1^A \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 |\ \alpha > \beta,\ \eta < 1\right\}, \tag{40}$$

$$\bar{\Theta}_1^B \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 |\ \alpha > \beta,\ \eta > 1\right\}, \tag{41}$$

$$\bar{\Theta}_1^C \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 |\ \alpha < \beta,\ \eta < 1\right\}, \tag{42}$$

$$\bar{\Theta}_1^D \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 |\ \alpha < \beta,\ \eta > 1\right\}, \tag{43}$$

$$\bar{\Theta}_2^A \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 |\ \alpha > \beta,\ \eta < 1\right\}, \tag{44}$$

$$\bar{\Theta}_2^B \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 |\ \alpha > \beta,\ \eta > 1\right\}, \tag{45}$$

$$\bar{\Theta}_2^C \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 |\ \alpha < \beta,\ \eta < 1\right\}, \tag{46}$$

$$\bar{\Theta}_2^D \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 |\ \alpha < \beta,\ \eta > 1\right\}. \tag{47}$$

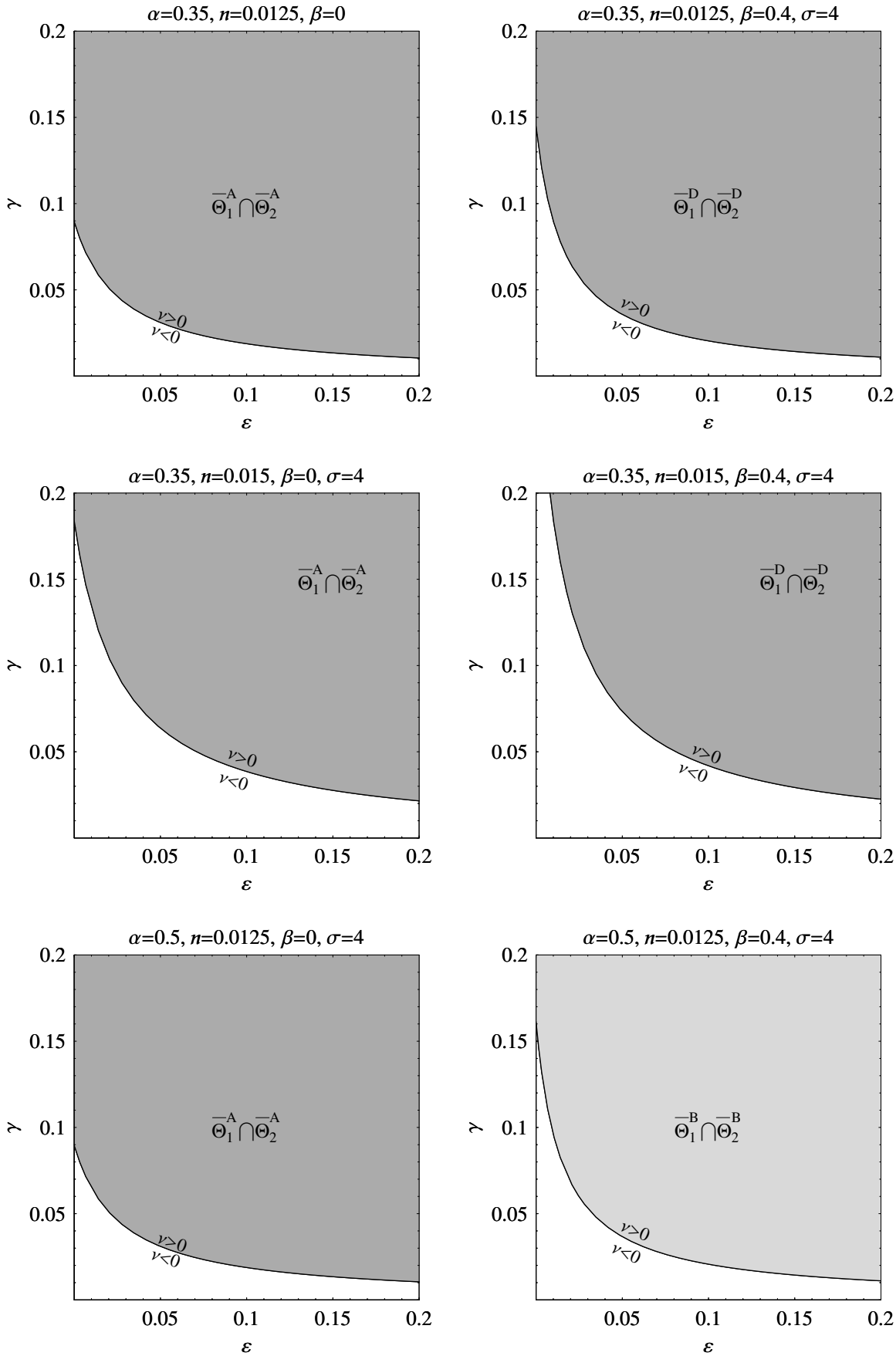

Figure 7: The parameter space for δ=0.065, κ=0.0265, ρ=0.03 and σ=4.

Proposition 6 *If the parameter values* $\bar{\theta} \in \left(\bar{\Theta}_1^A \cap \bar{\Theta}_2^A\right) \cup \left(\bar{\Theta}_1^D \cap \bar{\Theta}_2^D\right)$*, there is a neighborhood of the balanced growth path in which there exists a unique competitive equilibrium.*

Proof: Follows directly from Propositions 3 and 5. ■

In Figure 6, we set the values of $\delta = 0.065$, and $\rho = 0.03$, fix the intertemporal elasticity of substitution to σ=4, and vary the magnitudes of both external effects β and γ. The per-capita rate of output growth is set to $\kappa = 0.0265$, which approximates the average per-capita growth rates between 1997 and 2006 in the United Kingdom, at 0.0241; Australia, at 0.0243; Canada, at 0.0251; Chile, at 0.0264; Turkey, at 0.0267; Pakistan, at 0.0271; Spain, at 0.272; or Sweden, at 0.0280 (see Figure 5). In the upper two and lower two panels we set the population growth rate to $n = 0.0125$, the share of physical capital to $\alpha = 0.35$ in the upper and lower panels, and to $\alpha = 0.5$ in the lower two panels. In the middle two panels we set the rate of population growth to $n = 0.015$. In the panels on the left-hand side, we set the rate of depreciation in the human capital sector to $\varepsilon = 0$, and to $\varepsilon = 0.05$ in the panels on the right-hand side. What emerges in each of the six panels is that given this high rate of intertemporal substitution, interior balanced growth paths only emerge if there is at least some curvature in human capital production at the private level. How much curvature is required, depends directly on both the rate of depreciation in that sector and the population's growth rate, and inversely on the magnitude of returns to scale at the social level in the production sector. By contrast, the relative share of physical capital in the production process has only a small impact on the admissible range of parameters that support balanced growth, but once again substantially affects the model's dynamic behavior.

Consider the left-hand panels of Figure 6, where $\varepsilon = 0$. For both instances where $n = 0.0125$, balanced growth only emerges if the value of γ surpasses 0.0896, and then only if there are no external effects in the production sector. Raise the population growth rate to $n = 0.015$, and this threshold rises to 0.1840. Furthermore, in the absence of any depreciation in the human capital sector, the degree of concavity we must introduce to ensure the existence of balanced growth rises steeply, as we increase the size of β. Contrast this with the behavior of the model if we introduce a degree of depreciation in the human capital sector. First, the threshold value of γ drops precipitously, to only 0.0310 if $n = 0.0125$ and to 0.0637 if $n = 0.015$. Second, these thresholds no longer rise quite so dramatically as the values of β increase.

We can further see the trade-offs between concavity at the private level in the production of human capital, and the rate of depreciation in that sector, in the quasi-concave relationship between ε and γ in the panels of Figure 7 that correspond to the necessary condition for $\nu > 0$ in (33). Again, there is a striking contrast between the required degree of concavity or depreciation, or combination of both, that support interior balanced growth paths for $n = 0.0125$ and $n = 0.015$.

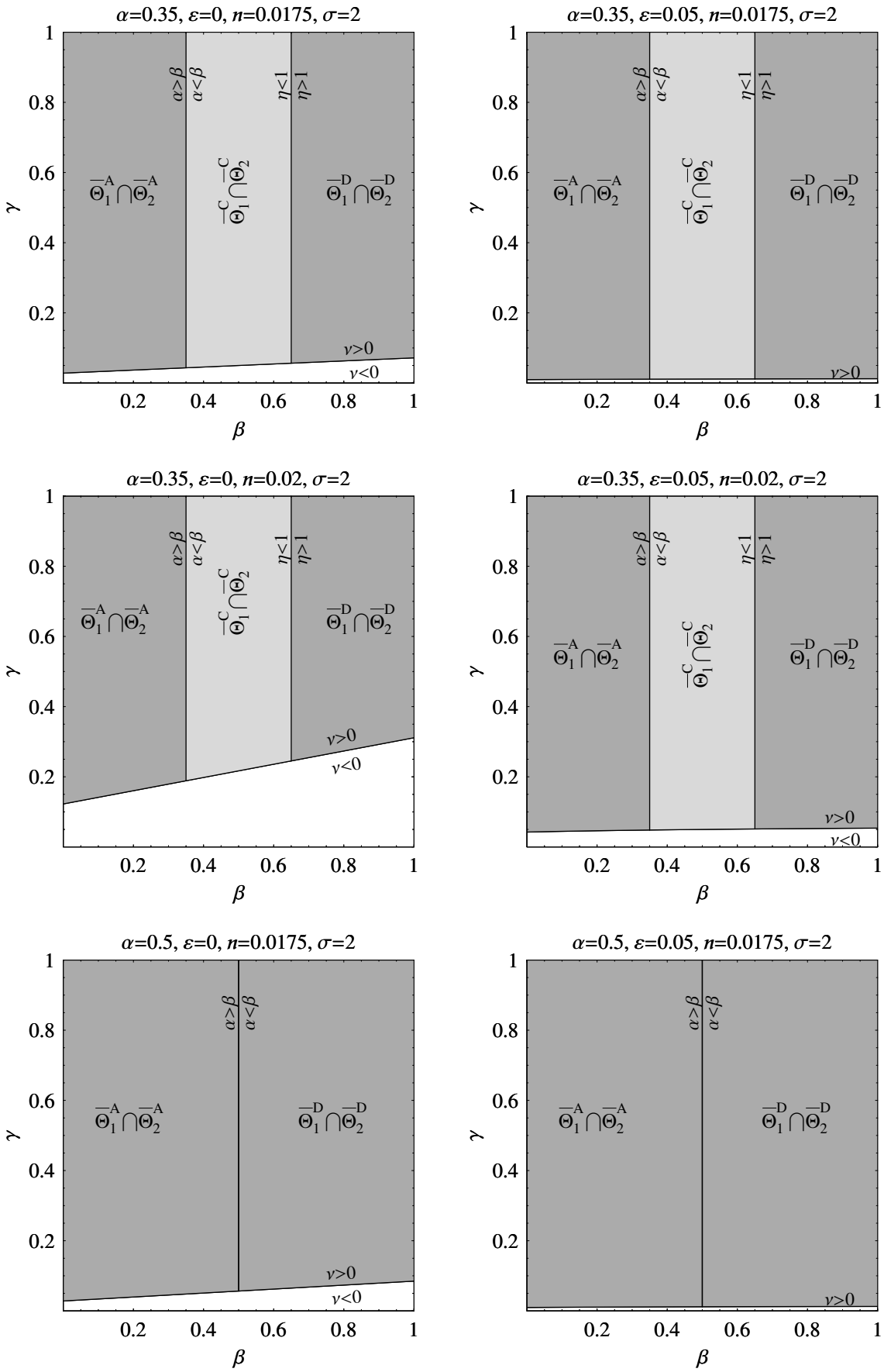

Figure 8: The parameter space for δ=0.065, κ=0.0265, ρ=0.03 and σ=2.

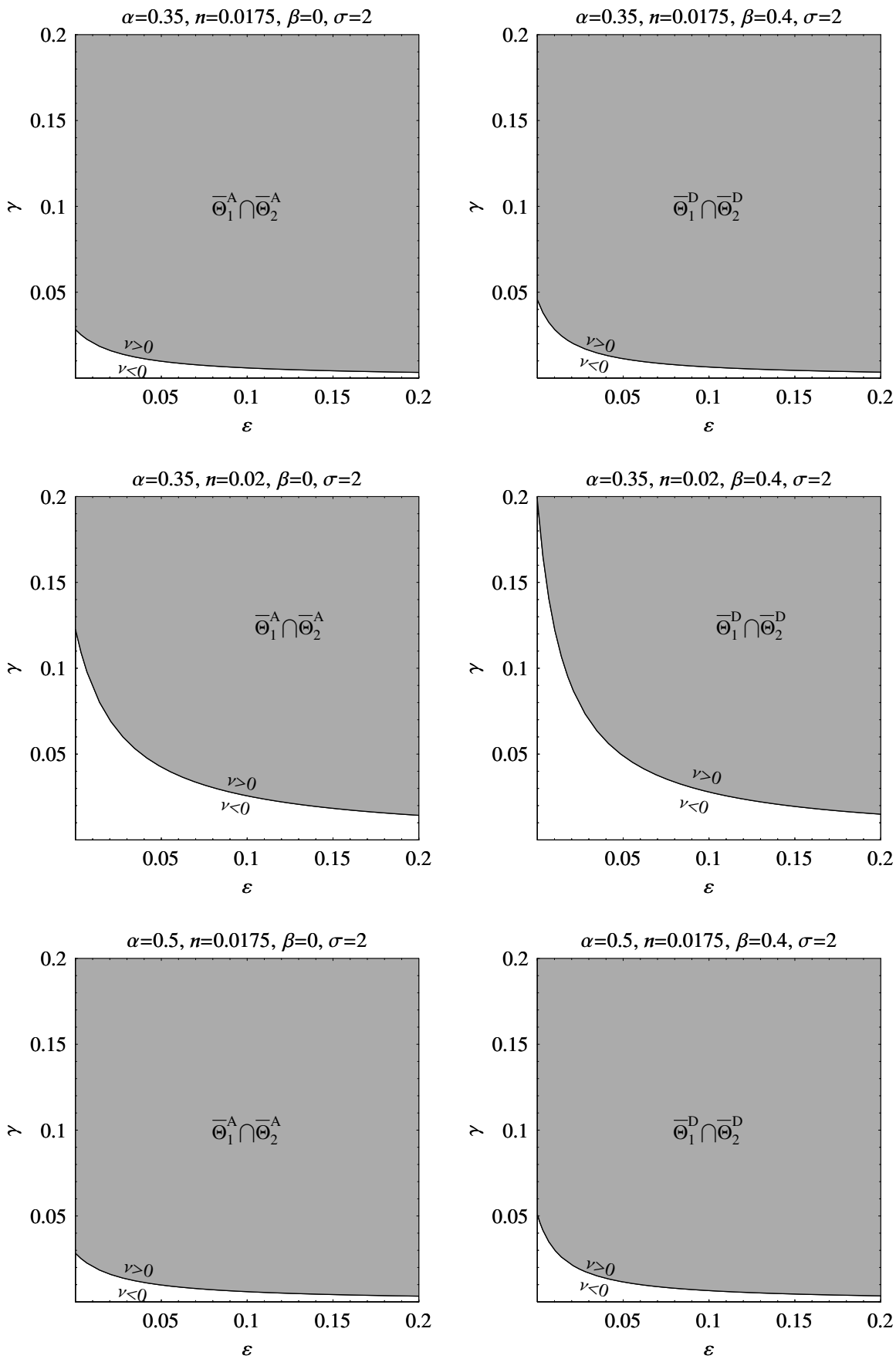

Figure 9: The parameter space for δ=0.065, κ=0.0265, ρ=0.03 and σ=2.

Rewriting the definition of $\bar{\Theta}_1$ in (37) in terms of a bound on the curvature parameter γ:

$$\bar{\Theta}_1 \equiv \left\{ \bar{\theta} \in \bar{\Theta} |\ \gamma > \frac{(1-\alpha+\beta)((n+\kappa-\rho)\sigma-\kappa)}{((1-\alpha+\beta)\varepsilon+(1-\alpha)\kappa)\sigma} \right\}, \tag{48}$$

if $\beta = 0$, the terms 1-α in the numerator and denominator of (48) cancel, and the parameter α no longer appears. Hence the corresponding panels in the uppermost and lowermost left-hand panels of Figure 6, are identical. However, if the value of β rises to 0.4, (48) does change slightly between the upper right and lower right panels of Figure 7, for different values of α. For $n = 0.0125$, and $\varepsilon = 0$, the lower limit for γ is 0.1448, if $\alpha = 0.35$; and 0.1613, if $\alpha = 0.5$. The differences for $\varepsilon = 0.05$ are smaller still, 0.0358 and 0.0369 for $\alpha = 0.35$, and $\alpha = 0.5$ respectively. It is in the local dynamic behavior of the model in the region of the balanced growth paths, rather than the conditions for the existence of the balanced growth path, where the value of α is decisive.

Returning to Figure 6, the set of parameters $\bar{\theta} \in \bar{\Theta}$ that support unique saddle path equilibria are confined to the subsets $\bar{\Theta}_1^A \cap \bar{\Theta}_2^A$ and $\bar{\Theta}_1^D \cap \bar{\Theta}_2^D$, and these areas are separated by the set $\bar{\Theta}_1^B \cap \bar{\Theta}_2^B$ that expands both to the left and right to cover a wider range of values for β, as α increases. In Figure 7, the regions of the parameter space that correspond to balanced growth paths as both γ and ε vary along the unit interval all correspond to $\bar{\Theta}_1^A \cap \bar{\Theta}_2^A$, if $\beta = 0$. By contrast if $\beta = 0.4$, and $\alpha = 0.35$, the relevant region corresponds to $\bar{\Theta}_1^D \cap \bar{\Theta}_2^D$. In either case, the addition of a combination of external effects and depreciation is sufficient to guarantee saddle path stable equilibria and unique interior balanced growth paths. Only if the share of capital in production exceeds the size of the external effect in that sector, and both are quite high ($\alpha = 0.5$, $\beta = 0.4$) do all interior balanced growth paths correspond to the parameter subspace $\bar{\Theta}_1^B \cap \bar{\Theta}_2^B$ in the lowermost right-hand panel of Figure 7. Here the parameters that correspond to balanced growth paths are associated with locally unstable dynamics—solutions that satisfy (5)—(8), but involve non-steady state ratios of physical to human capital will correspond to dynamic paths that violate (9) and (10).

Despite this last restriction the two-sector endogenous growth model does accommodate intertemporal elasticities of substitution at the upper bound of estimates we find in the empirical literature, and still generate valid balanced growth paths characterized by saddle path stable local dynamics for relatively high rates of population growth, provided the human capital accumulation process is augmented by small degrees of external effects and depreciation. In Figures 8 and 9 the intertemporal elasticity of substitution is set at $\sigma = 2$. Here the model can easily accommodate rates of population growth in the range of n=0.0175 to n=0.02, as long as the values of γ and ε are above relatively small thresholds.

In Figure 8 the range of parameter values covered by $\bar{\Theta}_1^A \cap \bar{\Theta}_2^A$ and $\bar{\Theta}_1^D \cap \bar{\Theta}_2^D$, that which supports steady state growth and saddle path dynamics, is slightly large than in Figure 6. In the panels in

the upper two rows the subspace that seperate them is $\bar{\Theta}_1^C \cap \bar{\Theta}_2^C$ rather than $\bar{\Theta}_1^B \cap \bar{\Theta}_2^B$, but again corresponds to growth paths charactorized by unstable dynamics. In the last row with $\alpha = 0.5$, $\bar{\Theta}_1^C \cap \bar{\Theta}_2^C$, is of measure zero, corresponding to the points where $\beta = 0.5$. Hence all the steady state growth paths in this case are generically saddle path stable, and all the areas corresponding to steady state growth in Figure 9 fall into the category of $\bar{\Theta}_1^A \cap \bar{\Theta}_2^A$ in the left hand side panels, or $\bar{\Theta}_1^D \cap \bar{\Theta}_2^D$ in the right hand side panels.

7 Conclusion

The Uzawa-Lucas two sector endogenous growth model accommodates two important observations: there are large differences in the rental rates for human capital (wage for a given skill level) across countries, and also differences between the growth rates of physical and human capital within each country. Hence arises the need to understand precisely what combinations of parameter values and steady state growth rates the model can and cannot accommodate.

Unfortunately, the usefullness of the model in its original form, is somewhat hampered by its inability to accommodate preferences characterized by high intertemporal elasticities of substitution, particularly if the rate of population growth is high as well. By including external effects and depreciation, we remedy this problem—the Uzawa-Lucas two sector endogenous growth model can accommodate a far wider range of parameterizations than previously thought. Hopefully, with this full set of necessary and sufficient conditions that guarantee the existence of unique interior balanced growth paths, and saddle-path stable local dynamics in hand, applied macroeconomic theorists will be able to put the model to use in many more contexts as well as extend it still further. Furthermore future studies in the empirics of growth now have a model more general and flexible to estimate and test.

8 Appendix

8.1 If $\alpha = \beta$

Define the subsets of $\{\Theta_1, \Theta_2\}$:

$$\begin{aligned}
\Theta_1^D &\equiv \{\theta \in \Theta_1 | n + (\eta - \gamma)\nu - \varepsilon\eta < \rho < \left(\frac{1 - \eta + \alpha\eta}{\alpha}\right)(n + \delta + (1 - \gamma)\nu - \varepsilon) \text{ and } \alpha = \beta\}, \\
\Theta_2^D &\equiv \{\theta \in \Theta_2 | \left(\frac{1 - \eta + \alpha\eta}{\alpha}\right)(n + \delta + (1 - \gamma)\nu - \varepsilon) < \rho < n + (\eta - \gamma)\nu - \varepsilon\eta \text{ and } \alpha = \beta\}, \\
\Theta_2^E &\equiv \{\theta \in \Theta_2 | \varepsilon < (1 - \alpha)(n + \delta) + (1 - \gamma)\nu \text{ and } \alpha = \beta\}.
\end{aligned}$$

Corollary 1 *If $\alpha = \beta$, the necessary and sufficient condition for the existence of an interior balanced growth path is: $\theta \in \Theta_1^D \cup \left(\Theta_2^D \cap \Theta_2^E\right)$.*

Proof: If $\theta \in \{\Theta_1, \Theta_2\}$ then $u^* \in (0,1)$. If $\alpha = \beta$, then $k^* = \frac{(1-(1-\alpha)\eta)(n+\delta+(1-\gamma)\nu-\varepsilon)-\alpha(\delta+\rho)}{(1-\alpha)(1-\eta)}$ and $c^* = \frac{(1-\alpha)(n+\delta)+(1-\gamma)\nu-\varepsilon}{\alpha}$. If $\theta \in \Theta_1^D \Rightarrow \eta < 1$ and $\widetilde{k}^* > 0$ iff $\rho < \left(\frac{1-\eta+\alpha\eta}{\alpha}\right)(n+\delta+(1-\gamma)\nu-\varepsilon)$. This in turn implies $(1-\alpha)(n+\delta) + (1-\gamma)\nu - \varepsilon > 0 \Rightarrow \widetilde{c}^* > 0$. If $\theta \in \Theta_2^D \Rightarrow \eta > 1$ and $k^* > 0$ iff $\rho > \left(\frac{1-\eta+\alpha\eta}{\alpha}\right)(n+\delta+(1-\gamma)\nu-\varepsilon)$. If $\theta \in \Theta_2^E \Rightarrow \eta > 1$ and $\widetilde{c}^* > 0$ iff $\varepsilon < (1-\alpha)(n+\delta)+(1-\gamma)\nu$. ■

8.2 The linearized dynamic system

The non-linear dynamic system is linearized:

$\mathbf{J}_{11} = \sigma\left(\alpha\widetilde{k}^{*\alpha-1} - \rho\right) - \vartheta\left(\frac{(1-\gamma)\nu}{\alpha} - 2\frac{\widetilde{c}^*}{\widetilde{k}^*}\right)$

$\mathbf{J}_{12} = \alpha(\alpha-1)\sigma\widetilde{k}^{*\alpha-1} - \vartheta\frac{\widetilde{c}^*}{\widetilde{k}^*}$

$\mathbf{J}_{21} = (\vartheta-1)\frac{\widetilde{c}^*}{\widetilde{k}^*}$

$\mathbf{J}_{22} = \alpha\widetilde{k}^{*\alpha-1} - \vartheta\frac{(1-\gamma)\nu}{\alpha}$

Along the Balanced Growth Path: $\sigma\left(\alpha\widetilde{k}^{*\alpha-1} - \rho\right) = \vartheta\left(\frac{(1-\gamma)\nu}{\alpha} - \frac{\widetilde{c}^*}{\widetilde{k}^*}\right)$, substituting $\vartheta = \frac{1-\alpha+\beta}{\alpha-\beta}\frac{\alpha}{1-\alpha}$ and rewriting in terms of parameters using

$\frac{\widetilde{c}^*}{\widetilde{k}^*} = \left[\rho - \frac{(1-\alpha+\beta)(\alpha-1/\sigma)}{(1-\alpha)(\alpha-\beta)}(1-\gamma)\nu\right]\frac{(\alpha-\beta)}{\alpha(1-\eta)}$ we get $\mathbf{J}$ in (30) in Section 4.

References

[1] R. A. Amano and T. S. Wirjanto, Intratemporal substitution and government spending, *Review of Economics and Statistics*, **79** (1997), 605-609.

[2] O. P. Attanasio and G. Weber, Intertemporal substitution, risk aversion and the euler equation for consumption, *Economic Journal*, **99** (1989), 59-73.

[3] R. Bansal and A. Yaron, Risks for the long run: A potential resolution of asset pricing puzzles, *Journal of Finance*, **59** (2004), 1481-1509.

[4] R. Bansal, D. Kiku and A. Yaron, Risks for the long run: Estimation and inference, Working Paper (2007).

[5] R. J. Barro and G. S. Becker, A reformulation of the economic theory of fertility, *Quarterly Journal of Economics* **103** (1988) 1-25.

[6] R. J. Barro and G. S. Becker, Fertility choice in a model of economic growth, *Econometrica* **57** (1989) 481-501.

[7] M. Ben-Gad, Fiscal policy and indeterminacy in models of endogenous growth, *J. Econ. Theory* **108** (2003) 322-344.

[8] J. Benhabib and R. Perli, Uniqueness and indeterminacy: Transitional dynamics in a model of endogenous growth, *J. Econ. Theory* **63** (1994) 113-142.

[9] J. Benhabib and B. Jovanovic, Externalities and growth accounting, *Am. Econ. Rev.* **81** (1991) 82-113.

[10] J. Caballe and M. S. Santos, On endogenous growth with physical and human capital, *J. Polit. Econ.* **101** (1993), 1042-1067.

[11] S. N. Durlauf, A. Kourtellos and C. M. Tang, Are any growth theories robust?, *Economic Journal*, **118** (2008), 329-346.

[12] M. Fuse, Estimating intertemporal substitution in Japan, *Applied Economics Letters* **11** (2004), 267-269.

[13] J. Gruber, A tax-based estimate of the elasticity of intertemporal substitution, National Bureau of Economic Reserach, Working Paper 11945 (2006).

[14] R. E. Hall, The relation between price and marginal cost in U.S. Industry, *J. Polit. Econ.* **96** (1988), 921-947.

[15] S. Hamori, Consumption growth and the intertemporal elasticity of substitution: some evidence from income quintile groups in Japan, *Applied Economics Letters* **3** (1996), 529-32.

[16] L. P. Hansen and K. J. Singleton, Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica* **50** (1982), 1269-1286.

[17] L. P. Hansen, J. Heaton, J. Lee and N. Roussanova, Intertemporal Substitution and Risk Aversion, Handbook of Econometrics Vol. 6a, (J. J. Heckman and E. E. Leamer Ed.), pp. 3967-4056, Elsevier North Holland, Amsterdam, 2007.

[18] S. G. Harrison, Evidence on the empirical plausibility of externalities and indeterminacy in a two sector model, Working Paper 98-05, Barnard College (1998).

[19] L. E. Jones, R. E. Manuelli, H. E. Siu and E. Stacchetti, Fluctuations in Convex Models of Endogenous Growth, I: Growth effects, *Review of Economic Dynamics* **8** (2005) 780-804.

[20] R. E. Lucas Jr., On the mechanics of economic development, *J. Monet. Econ.* **22** (1988), 3-42.

[21] C. Mulligan, Substitution over time: another look at life-cycle labor supply, in *NBER Macroeconomics Annual*, (B. S. Bernanke and J. J. Rotemberg Ed.), pp. 75-134, MIT Press, Cambridge MA, 1998.

[22] C. J. M. Paul and D. S. Siegel, Scale and industry agglomeration externalities: a dynamic cost function approach, *Am. Econ. Rev.* **89** (1999), 272-290.

[23] A. Tversky and D. Kahneman, Advances in prospect theory: Cumulative representation of uncertainty, *Journal of Risk and Uncertainty* 5 (1992), 297-323.

[24] A. Vissing-Jørgensen and O. P. Attanasio, Stock market participation, intertemporal substitution and risk aversion, *Am. Econ. Rev.* **93** (2003), 383-391.

[25] D. Xie, Divergence in Economic Performance: Transitional dynamics with multiple equilibria, *J. Econ. Theory* **63** (1994), 97-112.