Controlling National Debt Dynamics

A First Approach

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CONTROL THEORY IN ECONOMICS

Optimal Control	**Feedback Design**
Optimization of certain quantities over a time period	Stabilization Tracking
	Phillips (1954, 1957), Baumol (1961), Aoki (1975)

Optimal Control is widespread in Economics

Feedback design was involved but received a lot of critisism.

It has been recently restimulated, following the financial crisis.

Wieland and Westerhoff (2005), Athanasiou and Kotsios (2008), Athanasiou-Karafyllis-Kotsios (2008), He and Westerhoff (2005), Leventides-Kollias (2014), Leventides-Kotsios (2004).

The General Idea

We describe the dynamic evolution of the National Income and the National Debt my means of certain Discrete Time Models.

By considering some of the variables of the above models as inputs, we design feedback-laws which lead the system to a desired behaviour.

The Aspects of the Problem

- Economical Approach
- Mathematical Approach
  - Dynamics
  - Control

THE MODELS

- **The General Model.**

$$Y_t = \Phi(Y_{t-1}, Y_{t-2}, \ldots, G_t, G_{t-1}, \ldots, R_t, R_{t-1}, \ldots)$$

$$B_t = B_{t-1} + rB_{t-1} + G_t - \tau Y_{t-1}$$

where:

Y_t	The National Income	B_t	The National Debt
G_t	The Goverment Expenditure	R_t	Extra Factors
r	The interest rate	τ	The tax coefficient
t	The Discrete Time		

- **The Classical Linear Model**

$$\left.\begin{array}{l} Y_t = I_t + C_t + G_t \\ C_t = (1-s)Y^d_{t-1} + sY^d_{t-2} \\ Y^d_t = Y_t - T_t \\ T_t = \tau Y_{t-1} \\ I_t = v(Y_{t-1} - Y_{t-2}) \\ \\ B_t = B_{t-1} + rB_{t-1} + G_t - T_t \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + \\ +a_3 Y_{t-3} + \lambda_0 G_t \\ \\ B_t = (1+r)B_{t-1} + G_t - \tau Y_{t-1} \end{array}}$$

- $a_1 = 1 + v - s$
- $a_2 = s - v - \tau(1-s)$
- $a_3 = -\tau s$
- $\lambda_0 = 1$

- **An Extension of the Classical Linear Model**

$$Y_t = I_t + C_t + \lambda_0 G_t + \lambda_1 G_{t-1} + \lambda_2 G_{t-2}$$

$$C_t = (1-s)Y^d_{t-1} + sY^d_{t-2}$$

$$Y^d_t = Y_t - T_t, \quad T_t = \tau Y_{t-1}, \quad I_t = v(Y_{t-1} - Y_{t-2})$$

$$B_t = B_{t-1} + rB_{t-1} + G_t - T_t$$

$$\Downarrow$$

$$Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + +a_3 Y_{t-3} + \lambda_0 G_t + \lambda_1 G_{t-1} + \lambda_2 G_{t-2}$$

$$B_t = (1+r)B_{t-1} + G_t - \tau Y_{t-1}$$

- **A General Extension of the Classical Linear Model**

$$Y_t = I_t + C_t + \sum_{i=0}^{k} \lambda_i G_{t-i}$$

$$C_t = (1-s)Y_{t-1}^d + sY_{t-2}^d$$

$$Y_t^d = Y_t - T_t, \quad T_t = \tau Y_{t-1}, \quad I_t = v(Y_{t-1} - Y_{t-2})$$

$$B_t = B_{t-1} + rB_{t-1} + G_t - T_t$$

$$\Downarrow$$

$$Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + +a_3 Y_{t-3} + \textstyle\sum_{i=0}^{k} \lambda_i G_{t-i}$$

$$B_t = (1+r)B_{t-1} + G_t - \tau Y_{t-1}$$

- **A Model with Extra Taxation**

$$Y_t = I_t + C_t + \lambda_0 G_t + \lambda_1 G_{t-1} + \lambda_2 G_{t-2} + a_4 E_{t-1} + a_5 E_{t-2}$$

$$C_t = (1-s) Y^d_{t-1} + s Y^d_{t-2}$$

$$Y^d_t = Y_t - T_t, \quad T_t = \tau Y_{t-1}, \quad I_t = v(Y_{t-1} - Y_{t-2})$$

$$B_t = B_{t-1} + rB_{t-1} + G_t - T_t - E_t$$

$$\Downarrow$$

$$Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + + a_3 Y_{t-3} + \sum_{i=0}^{3} \lambda_i G_{t-i} + a_4 E_{t-1} + a_5 E_{t-2}$$

$$B_t = (1+r) B_{t-1} + G_t - \tau Y_{t-1} - E_t$$

- **A Nonlinear Extension of the Classical Model**

$$Y_t = I_t + C_t + \lambda_0 G_t + \lambda_1 G_{t-1} + \lambda_2 G_{t-2}$$

$$C_t = (1-s) Y^d_{t-1} + s Y^d_{t-2}$$

$$Y^d_t = Y_t - T_t, \quad T_t = \tau Y_{t-1}, \quad I_t = v(Y_{t-1} - Y_{t-2}) - v(Y_{t-1} - Y_{t-2})^3$$

$$B_t = B_{t-1} + rB_{t-1} + G_t - T_t$$

$$\Downarrow$$

$$\boxed{\begin{array}{l} Y_t = a_1^* Y_{t-1} + a_2^* Y_{t-2} + + a_3^* Y_{t-3} + a_4^*(Y_{t-1} - Y_{t-2})^3 + \\ + \lambda_0 G_t + \lambda_1 G_{t-1} + \lambda_2 G_{t-2} \\ \\ B_t = (1+r) B_{t-1} + G_t - \tau Y_{t-1} \end{array}}$$

- **Another Nonlinear Model**

$$Y_t = aY_{t-1} + \frac{bd(\overline{Y}-Y_{t-1})^3 - cd(\overline{Y}-Y_{t-1})}{1+(\overline{Y}-Y_{t-1})} + \lambda_0 G_t + \lambda_1 G_{t-1} + \lambda_2 G_{t-2}$$

$$B_t = (1+r)B_{t-1} + G_t - \tau Y_{t-1}$$

$\overline{Y}$ stands for the "theoretical" long run equilibrium output.

This model takes into consideration the sentiment of the investors. If they are optimistic (pessimistic) the economy performs well (badly). [**Westerhoff, 2006**].

THE PROBLEM

From the Dynamical point of view

- Choose a model
- Study the Dynamic behavior
- Extract Economical results

From the Control point of view

- Choose a model
- Choose the control functions
- Specify a desired behavior
- Design a feedback - law
- Take under consideration, if possible, special economic characteristics

SPECIAL ISSUES

- **Many Models**

  There many models. Each of them is focused to different aproaches.

- **The high complexity**

  The problem is of high complexity. Stochastic models must be also included.

- **The time issue**

  Some economic functions change at different time instants than others.

- **The time the results must be obtained**

  In economics we want immediate results. There are not "initial" points of a trajectory.

- **The choice of the control functions**

  It is not very clear what functions
  can be considered as control functions

- **The initial conditions issue**

  Zero initial conditions do not exist in economics.

- **Stability versus instability**

  Economists do not prefer stability.

- **What is tracking? can we approach a trajectory;**

  What is tracking in economics? Must we follow
  exactly a given trajectory or can we approach it?

- **Implementation of a desired behavior**

  How do we implement a perfect economic behavior?
  As a system or as a reference signal?

- **Feasibility of the Feeback**

  The feedback-laws must have an economic meaning.

- **The Lucas critic**

  Lucas said, that any application of a feedback-law, changes the parameters of the system.

Special forms of the general problem we have studied

• Find a causal feedback rule concerning government expenditures so that National Income matches a pre-described trajectory, and the Debt is close to a given target.

• Find a causal feedback rule concerning government expenditures so that the Debt matches a pre-described trajectory, and the National Income is close to a given target.

• Calculate feedback rules for the government expenditures and the EXTRA taxation which will match BOTH the Income and the Debt to pre-described paths.

• Repeat the above problems by estimating the original system using adaptive algorithms

The First Problem in details - The Model Matching Methodology

We shall work with the model:

$$Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + a_3 Y_{t-3} + \lambda_0 G_t + \lambda_1 G_{t-1} + \lambda_2 G_{t-2}$$

$$B_t = (1+r)B_{t-1} + G_t - \tau Y_{t-1}$$

G_t input, Y_t, B_t output

Step 1: By using the lag operator q defined as: $q^i f(t) = f(t-i)$
We rewrite it as follows:

$$A_{11} Y(t) = \Gamma_1 G(t)$$

$$A_{21} Y(t) + A_{22} B(t) = \Gamma_2 G(t)$$

where, A_{11}, A_{21}, A_{22}, Γ_1, Γ_2, are properly defined q-polynomials.

Step 2: By using the reference sequences $Y^*(t)$ and an "artificial" input $u_c(t)$ we construct the desired system:

$$A_{11}^d Y^*(t) = \Gamma_1^d u_c(t)$$

Since there are many such desired systems, we parameterized their family by $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$. We denote it by:

$$A_{11}^d(\theta) Y^*(t) = \Gamma_1^d(\theta) u_c(t)$$

Explanation: We want to calculate a feedback - law of the form:

$$RG(t) = Tu_c(t) - SY(t)$$

R, S, T unknown q-polynomials, so that that closed loop-system will be equal to the desired. This means:

$$RA_{11}Y(t) = R\Gamma_1 G(t) = \Gamma_1 RG(t) = \Gamma_1(Tu_c(t) - SY(t))$$

$$\Rightarrow (RA_{11} + \Gamma_1 S)Y(t) = \Gamma_1 T$$

$$\Rightarrow RA_{11} + \Gamma_1 S = A_{11}^d(\theta) \quad \text{and} \quad \Gamma_1 T = \Gamma_1^d(\theta)$$

Step 3: We solve the algebraic equations:

$$R(\theta)A_{11} + \Gamma_1 S(\theta) = A_{11}^d(\theta)$$

$$\Gamma_1 T(\theta) = \Gamma_1^d(\theta)$$

This is a polynomial Diophantine equation. The solutions are described as :

$$R(\theta) = R_0(\theta) + \Gamma_1 \cdot Q$$

$$S(\theta) = S_0(\theta) - A_{11} \cdot Q$$

R_0, S_0 a pair of "initial" solutions, Q an arbitrary q-polynomial

Step 4: By assuming specific value to Q, so that causality is satisfied, we construct the feedback-law:

$$R(\theta)G(t) = T(\theta)u_c(t) - S(\theta)Y(t)$$

Step 5: Construct the corresponding closed-loop system for public debt:

$$(R(\theta)A_{21} + \Gamma_2 S(\theta))Y^*(t) + R(\theta)A_{22}\hat{B}(t) = \Gamma_2 T(\theta)u_c(t)$$

Step 6: We calculate a value $\overline{\theta}$ such that

$$||\hat{B}(t) - B^*(t)|| \to minimum$$

Step 7: We get the requested feedback-law.

$$R(\overline{\theta})G(t) = T(\overline{\theta})u_c(t) - S(\overline{\theta})Y(t)$$

A Simulation

The Greek Case.

Using econometric results, interest rate on public debt equal to 4%, and tax rate equal to 40%, we get for the Greek Case:

$$Y(t) - 0.056Y(t-1) + 0.42Y(t-2) + 0.18Y(t-3) = 0.59G(t) +$$

$$+0.77G(t-1) + 0.56G(t-2)$$

$$0.4Y(t) + B(t) - 1.04B(t-1) = 1.04B(t-1) + G(t) - 0.4Y(t)$$

A low-growth scenario

	Reference Sequences		Actual Values			Model Values			
	Y^*	B^*	Y	B	G	Y	B	G	B/Y
2010Q1	60187	298935	52766,6	309676	27472	60187	305327.6	29939.08	
2010Q2	60337	298188	56698,7	314140	27424	60337	311250	30066.95	
2010Q3	60488	297443	57796,4	322978	26725	60488	317477.3	30184.76	
2010Q4	60638	296699	54710	329515	32688	60638	324015.3	30303.6	1,34

(all values are in millions of euro)

Actual $B/Y = 1.58$

With Feedback- law

$$G(t) = 0.18Y(t-1)+0.045Y(t-2)-0.14G(t-1)-0.037G(t-2)+$$

$$+0.026u_c(t-1)+0.027u_c(t-2)$$

Future Research

- Design of Feedback laws with Adaptive Techniques
- Feedback design for Nonlinear systems
- Dynamics of Complex Nonlinear Systems
- New Models so that more complex economical aspects to be included (Stochastic Models)

Some Relevant Papers

S.Kotsios and J.Leventidis A FEEDBACK POLICY FOR A MODIFIED SAMUELSON-HICKS MODEL. International Journal of Systems Science, vol 35, (6), pp 331-341, 2004.

G. Athanasiou I. Karafyllis - S.Kotsios PRIZE STABILIZATION USING BUFFER STOCKS. Journal of Economic Dynamics and Control 32 (2008) 12121235

George Athanasiou, Stelios Kotsios AN ALGORITHMIC APPROACH TO EXCHANGE RATE STABILIZATION. Economic Modelling 25 (2008) 12461260.

Ilias Kostarakos-Stelios Kotsios, Fiscal Policy, Linear Feedback Control and Debt Stabilization, 1st International Conference in Economics and Business, Hellenic Open University, 6-7 February 2015, Athens, Greece.

Ilias Kostarakos-Stelios Kotsios, Fiscal Policy by using Linear Feedback Laws for Debt Stabilizing, 1st AMEF Conference, 6-7 April 2015, Thessaloniki, Greece.

Ilias Kostarakos-Stelios Kotsios, An Algorithmic Linear Feedback Approach to the Design of Fiscal Policy: The Cases of Greece and Cyprus, 7th Biennial PhD Symposium on Contemporary Greece and Cyprus, London School of Economics (LSE), 4-5 June 2015, London, UK.

Ilias Kostarakos-Stelios Kotsios, AN ALGORITHMIC LINEAR FEEDBACK APPROACH TO THE DESIGN OF FISCAL POLICY: THE CASE OF GREECE, Under Reviewing at Empirica. Journal of European Economics