Resource Allocation for System Robustness

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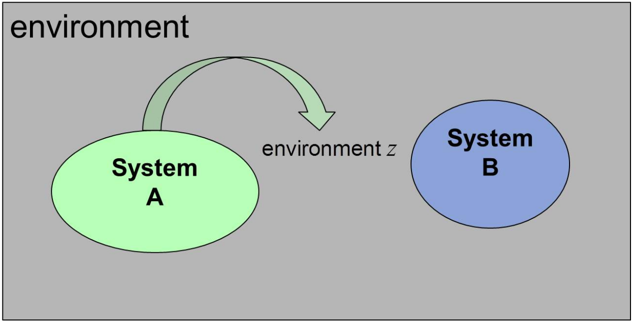
Outline

The talk pulls together a couple of longstanding ideas to develop a method for the improvement of system performance.
The overall aim is to reduce variation in performance by controlling the design parameters.

- Structure functions - provide an algebraic representation of the system
- Path and cut sets
- Identification of component/subsystem importance
- Use of Taguchi methods to design improvements

sources of variation

- Manufacturing variation
- Different suppliers
- Variations in Use
- System communication



Ashby and variation.

State Representation

Represent the system as a binary vector

$$\mathbf{X} = [x_1, x_2, \ldots, x_n]$$

where $x_i = 1$ denotes a working element and $x_i = 0$ denotes a failed component.

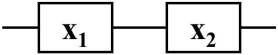
Definition 1 (Structure Function)

$\phi(\mathbf{X}) = 1$ path through system

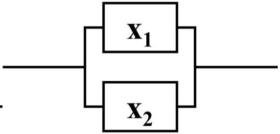
and

$\phi(\mathbf{X}) = 0$ no path through system

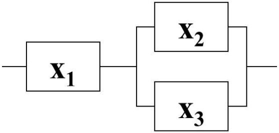
simple system block diagrams



(a) series



(b) parallel



(c) series-parallel

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definitions

Definition 2 (Path Set)

A path set $\mathcal{P}$ is a set of elements. The system contains a path through it if all elements in the set $\mathcal{P}$ work.

Definition 3 (Cut Set)

A cut set $\mathcal{C}$ is a set of elements. There is no path through the system if all elements in $\mathcal{C}$ fail.

Definition 4 (Minimal Path Set)

A path set $\mathcal{P}$ is minimal if it ceases to be a path set with the removal of a single element.

Definition 5 (Minimal Cut Set)

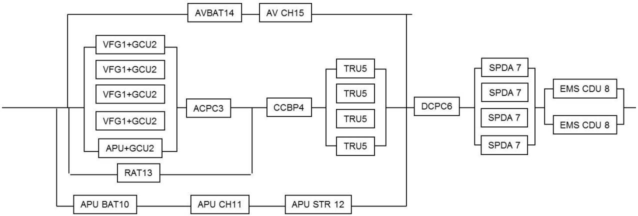
A cut set $\mathcal{C}$ is minimal if it ceases to be a cut set with the removal of a single element.

simple structures

System $\mathbf{X} = [x_1, x_2]$

- Series system $\phi(\mathbf{X}) = x_1 \times x_2$
- Parallel system $\phi(\mathbf{X}) = 1 - (1 - x_1) \times (1 - x_2)$
- minimal cut set representation $\phi(\mathbf{X}) = \prod_{i=1}^{n} \mathcal{C}_i$
- minimal path set representation $\phi(\mathbf{X}) = 1 - \prod_{i=1}^{n} [1 - \mathcal{P}_i]$

A complex avionics system example



$$\phi_1 = [(1-(1-x_1)\cdot(1-x_{14}))\cdot x_2]$$
$$\phi_2 = 1-(1-\phi_1)\cdot(1-x_{11})\cdot x_3\cdot x_4$$
$$\phi_3 = 1-(1-x_{12}\cdot x_{13})\cdot(1-\phi_2)\cdot(1-x_8\cdot x_9\cdot x_{10})\,;$$
$$\phi = \phi_3\cdot x_5\cdot x_6\cdot x_7$$

$$\phi = 1-(1-x_{12}\cdot x_{13})\cdot(1-\{1-(1-[(1-(1-x_1)\cdot(1-x_{14}))\cdot x_2])\cdot(1-x_{11})\cdot x_3$$

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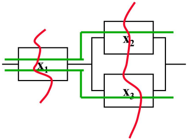
Three element system



x_1	x_2	x_3	ϕ
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
0	0	0	0

Table: structure function

	set elements	state vector
path set	$\{1,2,3\}$	$[1,1,1]$
	$\{1,2\}$	$[1,1,0]$
	$\{1,3\}$	$[1,0,1]$
cut sets	$\{1,2,3\}$	$[0,0,0]$
	$\{1,2\}$	$[0,0,1]$
	$\{1,3\}$	$[0,1,0]$
	$\{2,3\}$	$[1,0,0]$
	$\{1\}$	$[0,1,1]$

Table: Path and Cut

The reliability can be written as

$$\mathbf{r}(t) = \{r_1(t), r_2(t), \ldots, r_m(t)\}, \; r_i(t) = \mathbf{E}[x_i(t)]$$
$$\mathcal{R}(\mathbf{r}(t)) = \mathbf{E}[\phi(\mathbf{X}(t))] = \phi(\mathbf{E}[\mathbf{X}(t)]) = \phi(\mathbf{r}(t))$$

$$\mathbf{r}(t) = \{r_1(t), r_2(t), \ldots, r_m(t)\}.$$

Pivotal decomposition

$$\phi[\mathbf{X}(t)] = x_i(t)\phi[1_i, \mathbf{X}(t)] + (1 - x_i(t))\,\phi[0_i, \mathbf{X}(t)]$$

$$\mathcal{R}(\mathbf{r}(t)) = r_i(t)\mathcal{R}[1_i, \mathbf{r}(t)] + (1 - r_i(t))\,\mathcal{R}[0_i, \mathbf{r}(t)]$$

$$\mathcal{R}(\mathbf{r}(t)) = r_i(t)\{r_j(t)\mathcal{R}[1_i, 1_j, \mathbf{r}(t)] + (1 - r_j(t)\mathcal{R}[1_i, 0_j, \mathbf{r}(t)]\} + \cdots$$
$$\cdots (1 - r_i(t))\{r_j(t)\mathcal{R}[0_i, 1_j, \mathbf{r}(t)] + (1 - r_j(t)\mathcal{R}[0_i, 0_j, \mathbf{r}(t)]\}$$

Birnbaum's importance

- Birnbaum's importance $\mathbf{B}_i[\mathbf{r}(t)] = \dfrac{\partial \mathcal{R}(\mathbf{r}(t))}{\partial r_i(t)}$
- pivotal decomposition

$$\mathbf{B}_i[\mathbf{r}(t)] = \frac{\partial \mathcal{R}(\mathbf{r}(t))}{\partial r_i(t)} = \mathcal{R}\left[1_i, \mathbf{r}(t)\right] - \mathcal{R}\left[0_i, \mathbf{r}(t)\right]$$

-

$$\mathbf{C}_{i,j}[\mathbf{r}(t)] = \frac{\partial^2 \mathcal{R}(\mathbf{r}(t))}{\partial r_i(t) \partial r_j(t)}$$

$$\mathbf{C}_{i,j} = \left\{\mathcal{R}\left[1_i, 1_j, \mathbf{r}(t)\right] - \mathcal{R}\left[1_i, 0_j, \mathbf{r}(t)\right]\right\} - \left\{\mathcal{R}\left[0_i, 1_j, \mathbf{r}(t)\right] - \mathcal{R}\left[0_i, 0_j, \mathbf{r}(t)\right]\right\}$$

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Importance

The three component system

$(\cdot, x_2, x_3)$	$\phi(1, x_2, x_3) - \phi(0, x_2, x_3)$	critical paths for 1	
$(\cdot, 0, 0)$	0		
$(\cdot, 0, 1)$	1	$[1, 3]$	$\mathbf{B}_1[\mathbf{r}(t)] = \frac{3}{4}$
$(\cdot, 1, 0)$	1	$[1, 2]$	
$(\cdot, 1, 1)$	1	$[1, 2, 3]$	

$(x_1, \cdot, x_3)$	$\phi(x_1, x_1, x_3) - \phi(x_1, 0, x_3)$	critical paths for 2	
$(0, \cdot, 0)$	0		
$(0, \cdot, 1)$	0		$\mathbf{B}_2[\mathbf{r}(t)] = \frac{1}{4}$
$(1, \cdot, 0)$	1	$[1, 2]$	$\mathbf{B}_3[\mathbf{r}(t)] = \frac{1}{4}$
$(1, \cdot, 1)$	0		

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Improvement

- Improvement potential $\mathbf{I}(\mathbf{i}) = \mathcal{R}(\mathbf{1_i}, \mathbf{r}) - \mathcal{R}(\mathbf{r})$
- an m components system
- Let the nominal reliability be $\mathbf{r}_0(t) = \{r_{0,1}(t), r_{0,2}(t), \ldots, r_{0,m}(t)\}$ and assume $\mathbf{E}\left[\mathbf{r}(t)\right] = \mathbf{r}_0(t)$.
- Increment in reliability is $\Delta\mathcal{R} = \mathcal{R}(\mathbf{r}) - \mathcal{R}(\mathbf{r}_0)$
- Taylor expansion

$$\mathcal{R}(\mathbf{r}) - \mathcal{R}(\mathbf{r}_0) = \sum_{i=1}^{m} \left.\frac{\partial\mathcal{R}(\mathbf{r})}{\partial r_i}\right|_{r=r_0} (r_i - r_{0,i}) = \sum_{i=1}^{m} \mathbf{B}_i[\mathbf{r}_0(t)]\,(r_i - r_{0,i})$$

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Design parameters

Assume reliability parameters are subject to variation
Variance of r_i is σ_i^2

$$\sigma_{\mathcal{R}}^2 = \sum_{i=1}^{m} \left(\frac{\partial \mathcal{R}(\mathbf{r})}{\partial r_i} \right)^2 \sigma_{r_i}^2$$

write in terms of Birnbaum importance

$$\sigma_{\mathcal{R}}^2 = \sum_{i=1}^{m} \mathbf{B}_i^2(\mathbf{r}) \sigma_{r_i}^2$$

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factors in variability

explanatory variables $\mathbf{z} = \{\zeta_1, \zeta_2, \ldots, \zeta_n\}$
where $\mathbf{r}_z(t) = \mathbf{r}(t \,|\, \mathbf{z})$
the first order approximation

$$\frac{\partial \mathcal{R}}{\partial \zeta_k} = \sum_{i=1}^{m} \frac{\partial \mathcal{R}}{\partial r_i} \frac{\partial r_i}{\partial \zeta_k} = \sum_{i=1}^{m} \mathbf{B}_i[\mathbf{r}(t)] \frac{\partial r_i}{\partial \zeta_k}$$

$$\frac{\partial^2 \mathcal{R}}{\partial \zeta_k \partial \zeta_\ell} = \sum_{i=1}^{m} \left\{ \frac{\partial \mathcal{R}}{\partial r_i} \frac{\partial^2 r_i}{\partial \zeta_k \partial \zeta_\ell} + \sum_{j=1}^{m} \frac{\partial^2 \mathcal{R}}{\partial r_i \partial r_j} \frac{\partial r_i}{\partial \zeta_k} \frac{\partial r_i}{\partial \zeta_\ell} \right\}$$

$$= \sum_{i=1}^{m} \left\{ \mathbf{B}_i[\mathbf{r}(t)] \frac{\partial^2 r_i}{\partial \zeta_k \partial \zeta_\ell} + \sum_{j=1}^{m} \mathbf{C}_{i,j}[\mathbf{r}(t)] \frac{\partial r_i}{\partial \zeta_k} \frac{\partial r_i}{\partial \zeta_\ell} \right\}$$

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expansion

Nominal values
$\mathbf{z}^{(0)} = \left\{ \zeta_1^{(0)} \zeta_2^{(0)}, \ldots, \zeta_n^{(0)} \right\}, \; \mathbf{r}^{(0)}(t) = \mathbf{r}(t \,|\, \mathbf{z}^{(0)}), \; \mathbf{B}\left[\mathbf{r}^{(0)}(t)\right] = \mathbf{B}_i^{(0)}$
increment $\Delta\zeta_k = \zeta_k - \zeta_k^{(0)}$.
Put

$$r_{i,k} = \frac{\partial r_i}{\partial \zeta_k}, \quad r_{i,k}^{(0)} = \left. \frac{\partial r_i}{\partial \zeta_k} \right|_{\zeta_k = \zeta_k^{(0)}},$$

A delta approximation gives

$$\Delta\mathcal{R} = \sum_{i=1}^{m} \mathbf{B}_i[\mathbf{r}(t)]\Delta r_i, \; \Delta r_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial \zeta_k} \Delta\zeta_k$$

$$\Delta\mathcal{R} = \sum_{i=1}^{m} \mathbf{B}_i^{(0)} \sum_{k=1}^{n} r_{i,k}^{(0)} \Delta\zeta_k$$

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Variances

Taking expectations $\mathbf{V}[\mathcal{R}] \cong \mathbf{E}[\Delta\mathcal{R}]^2$

$$\sigma_{\mathcal{R}}^2 = \mathbf{V}[\mathcal{R}] \cong \sum_{k=1}^{n} \left(\frac{\partial \mathcal{R}}{\partial \zeta_k}\right)^2 \sigma_{\zeta_k}^2 = \sum_{k=1}^{n} \mathbf{B}_i^2[\mathbf{r}(t)] \left(\frac{\partial r_i}{\partial \zeta_k}\right)^2 \sigma_{\zeta_k}^2$$

$$\sigma_{r_i}^2 = \sum_{j=1}^{n} \left(\frac{\partial r_i}{\partial \zeta_j}\right)^2 \sigma_{\zeta_j}^2 + 2 \sum_{j=1}^{n} \sum_{k>j}^{n} \frac{\partial r_i}{\partial \zeta_j} \frac{\partial r_i}{\partial \zeta_k} \mathrm{Cov}\left(\zeta_j, \zeta_k\right)$$

Assuming independence and dropping the term cross product terms

$$\sigma_{\mathcal{R}}^2 = \sum_{i=1}^{m} \mathbf{B}_i^2(t) \sigma_{r_i}^2 = \sum_{i=1}^{m} \mathbf{B}_i^2(t) \sum_{j=1}^{n} \left(\frac{\partial r_i}{\partial \zeta_j}\right)^2 \sigma_{\zeta_j}^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{B}_i^2(t) \left(\frac{\partial r_i}{\partial \zeta_j}\right)^2 \sigma_{\zeta_j}^2$$

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explicit models

- Designed experiments leading to linear models

$$\sigma^2_{r_i} = \sum_{j=1}^{n} \alpha_j \sigma^2_{\zeta_j}, \ \sigma^2_{\mathcal{R}} = \sum_{i=1}^{m} \mathbf{B}^2_i(\mathbf{r})\sigma^2_{r_i}$$

- Proportional hazards models

$$r_i = r_{i,0}^{\psi(\mathbf{z})}$$

  The derivatives

$$\frac{\partial r_i}{\partial \zeta_j} = \psi_j(\mathbf{z}) \ln\left[r_{i,0}\right] r_{i,0}^{\psi(\mathbf{z})}$$

- Accelerated failure time models

$$r_i(t) = r_{i,0}(\psi(\mathbf{z})t)$$

  The derivatives

$$\frac{\partial r_i}{\partial \zeta_j} = \left[\frac{\partial \psi(\mathbf{z})}{\partial \zeta_j} t\right] r_{i,0}\left(\psi(\mathbf{z})t\right)$$

An optimization problem

Assume for simplicity that the variances are to be reduced so that $\sigma_{r_i}^2 \mapsto \sigma_{r_i}^2 - s_i^2$. This can be formulated as a programming problem with

$$\min \sum_{i=1}^{m} \mathbf{B}_i^2(\mathbf{r}) \left(\sigma_{r_i}^2 - s_i^2\right) \tag{1}$$

subject to (2)

$$s_i^2 < \sigma_i^2 \ , i = 1 : m \tag{3}$$

$$\sum_{i=1}^{m} c_i s_i^2 \leq K \tag{4}$$

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