Consistent valuations with bilateral counterparty risk and funding

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Overview

Intro

Early 2000s ...

Intro

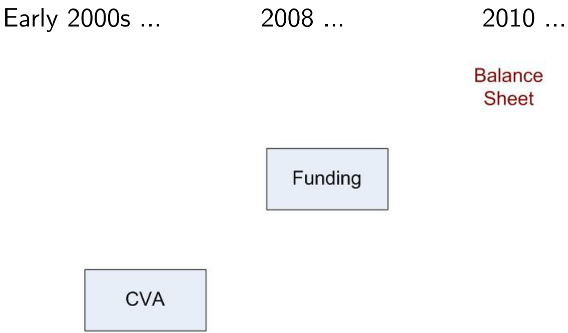
Early 2000s ...

2008 ...

Funding

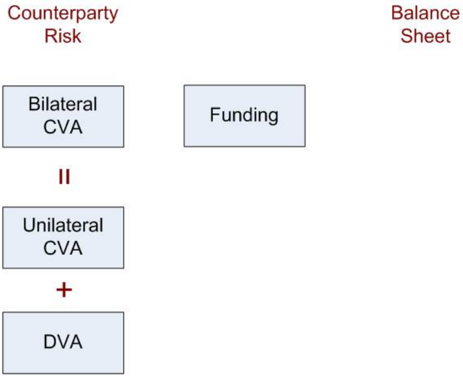
CVA

Intro



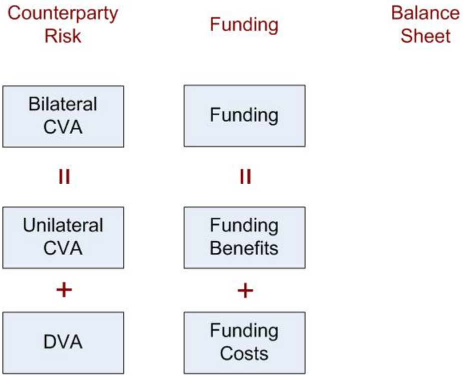
Intro

CVA has gone bilateral ...



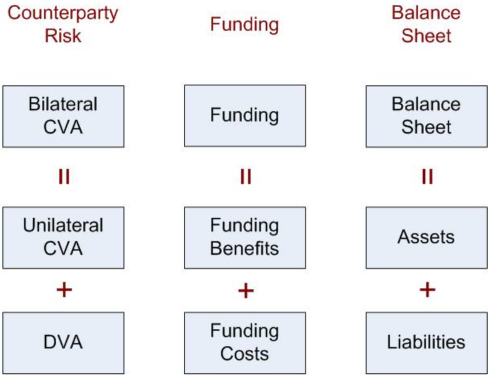
Intro

Funding: there are benefits, there are costs ...



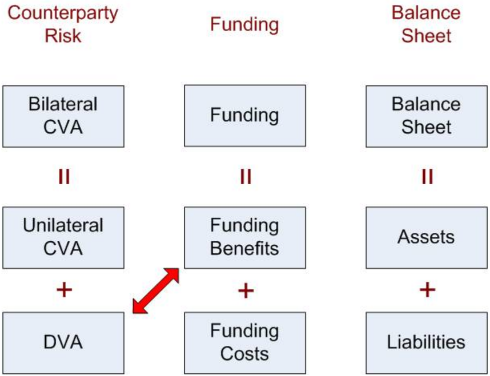
Intro

Balance sheet has assets and liabilities ... a lot of boxes ...



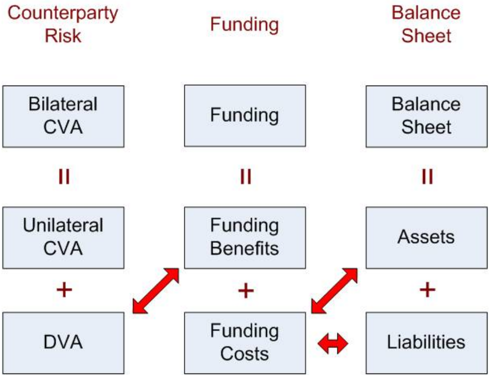
Intro

… but they are related - DVA and Funding Benefits



Intro

... and FCA is related to balance sheet.



Framework: replication

Extend BS PDE to include:

- bilateral counterparty risk
- funding costs.

Set up portfolio:

- $\hat{V}(S, t, J_B, J_C)$: derivative value (to issuer)
- J_B, J_C: default indicators
- S: underlying stock
- P_C: counterparty risky bond (zero recovery)
- P_B: issuer risky bond (zero recovery)
- β: cash accounts

Carefully consider funding of all positions

Derivative value at default

Default of counterparty or issuer:

- Claim based on derivative mark-to-market value M.
- ISDA master agreement seems to suggest $M = V$, where V is the risk-less derivative value.
- Then:

$$\begin{aligned} \hat{V}(t,S,1,0) &= V^+(t,S) + R_B V^-(t,S) \quad \text{B defaults first} \\ \hat{V}(t,S,0,1) &= R_C V^+(t,S) + V^-(t,S) \quad \text{C defaults first} \end{aligned}$$

- Brigo and Morini [1] discuss alternatives to this assumption.
- Burgard and Kjaer [2] also consider the case $M = \hat{V}$.

Processes

Asset price dynamics:

- For simplicity assume deterministic credit.

$$
\begin{aligned}
\frac{dS}{S} &= \mu dt + \sigma dW \\
\frac{dP_C}{P_C} &= r_C dt - dJ_C \\
\frac{dP_B}{P_B} &= r_B dt - dJ_B
\end{aligned}
$$

- r_B and r_C: yields of the risky (zero recovery) bonds.
- Assumptions:
  - asset S unaffected by defaults of B or C.
  - two independent Poisson processes J_B and J_C.

Replication strategy

Replication of risky derivative:

- $-\hat{V}(t) = \Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t)$
- Hedge out own credit risk: buy back α_B own bonds
- Hedge out counterparty credit risk: go short α_C counterparty bonds

Impose self-financing:

- $-d\hat{V}(t) = \delta(t)dS(t) + \alpha_B(t)dP_B(t) + \alpha_C(t)dP_C(t) + d\beta(t)$

Cash management

Decompose change in cash $d\beta(t)$:

- $d\beta(t) = d\beta_S(t) + d\beta_C(t) + d\beta_F(t)$

Funding of share position:

- $d\beta_S(t) = \delta(t)(\gamma_S(t) - q_S(t))S(t)dt$
- γ_S: dividend income
- q_S: net share position financing costs

Funding of counterparty bond position (short):

- $d\beta_C(t) = -\alpha_C(t)r(t)P_C(t)dt$
- short counterparty bond through repo

Cash management, cont.

Remaining cash position after purchase of own bonds:

- $d\beta_F(t) = r(t)(-\hat{V} - \alpha_B P_B)^+ dt + r_F(t)(-\hat{V} - \alpha_B P_B)^- dt$

If **positive**: invest in risk-free assets.

- don't add own credit risk (we just hedged it)

If **negative**: need to fund via external funding provider.

- costs r_F.
- $r_F = r$ if the derivative **can** be posted as collateral.
- $r_F = r + s_F$ with $s_F > 0$ if the derivative **cannot** be posted as collateral.

Replication strategy, cont.

The replication strategy then becomes:

$$
\begin{aligned}
-d\hat{V} &= \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\beta_S + d\beta_C + d\beta_F \\
&= \Big\{ - r\hat{V} + s_F(-\hat{V} - \alpha_B P_B)^- + (\gamma_S - q_S)\delta S \\
&\quad + (r_B - r)\alpha_B P_B + (r_C - r)\alpha_C P_C \Big\} dt \\
&\quad - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C + \delta dS.
\end{aligned}
$$

Replication strategy, cont.

On the other hand, by Ito's lemma:

$$d\hat{V} = \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C,$$

with

$$\Delta \hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0), \qquad \Delta \hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

so we can eliminate the risk factors by choosing:

$$\begin{aligned} \delta &= -\partial_S \hat{V}, \\ \alpha_B &= \frac{\Delta \hat{V}_B}{P_B} \\ \alpha_C &= \frac{\Delta \hat{V}_C}{P_C} \end{aligned}$$

PDE

Resulting PDE:

$$
\begin{aligned}
\partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} &= s_F(\hat{V} + \Delta\hat{V}_B)^+ - \lambda_B \Delta\hat{V}_B - \lambda_C \Delta\hat{V}_C \\
&= s_F V^+ - (R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}_t V &\equiv \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (q_S - \gamma_S) S \partial_S V \\
s_F &\equiv r_F - r \\
\lambda_B &\equiv r_B - r \\
\lambda_C &\equiv r_C - r
\end{aligned}
$$

Hedge positions

Asset	Pre default	Post Cparty default	Post Issuer default
S	$-\partial_S \hat{V}$	$-\partial_S \hat{V}$	$-\partial_S \hat{V}$
β_S	$\partial_S \hat{V}$	$\partial_S \hat{V}$	$\partial_S \hat{V}$
P_C	$-(1-R_C)V^+ - U$	0	$-(1-R_C)V^+ - U$
β_C	$(1-R_C)V^+ + U$	$(1-R_C)V^+ + U$	$(1-R_C)V^+ + U$
P_B	$-(1-R_B)V^- - U$	$-(1-R_B)V^- - U$	0
Deposit @r	$-R_B V^-$	$-R_B V^-$	$-R_B V^-$
I: borrow @r	$-V^+$	$-V^+$	$-V^+$
II: borrow @r_F	$-V^+$	$-V^+$	$-R_B V^+$
Total hedge (I)	$-(V+U)$	$-(V^- + R_C V^+)$	$-(R_B V^- + V^+)$
Total hedge (II)	$-(V+U)$	$-(V^- + R_C V^+)$	$-R_B V$
Derivative	$V+U$	$V^- + R_C V^+$	$R_B V^- + V^+$
Total $\hat{V} + \Pi$ (I)	0	0	0
Total $\hat{V} + \Pi$ (II)	0	0	$(1-R_B)V^+$

- Case I: Derivative **can** be posted as collateral so $s_F = 0$.
- Case II: Derivative **cannot** be posted as collateral so $s_F > 0$.

Credit and funding adjustments

The Ansatz $\hat{V} = V + U$ gives the following PDE for U:

$$\begin{aligned} \partial_t U + \mathcal{A}_t U - (r + \lambda_B + \lambda_C) U &= (1 - R_B)\lambda_B V^- \\ &\quad + (1 - R_C)\lambda_C V^+ \\ &\quad + s_F V^+ \\ U(T, S) &= 0 \end{aligned}$$

Solution by the Feynman-Kac theorem:

$$\begin{aligned} U(t, S) &= -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t \left[V^+(u, S(u)) \right] du \\ &\quad - (1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t \left[V^-(u, S(u)) \right] du \\ &\quad - \int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t \left[V^+(u, S(u)) \right] du \end{aligned}$$

Credit and funding adjustments (ctd)

Decompose U further as $U = CVA + DVA + FCA$ with

- **CVA**: (modified unilateral) credit value adjustment
- **DVA**: debt value adjustment
- **FCA**: funding **cost** adjustment

Some comments:

- **CVA**: is modified as it is conditioned to issuer not defaulting first
- **DVA**: excess earned by issuer when buying back own bonds **out of the positive cash account**
- thus this is a funding benefit that monetises "own counterparty risk" without defaulting
- **FCA**: cost if issuer has to use unsecured funding for negative cash account

Case I: $s_F = 0$

If the derivative **can** be used as collateral:

- perfect hedging in all scenarios *including* own default
- $s_F = 0$ and, therefore, FCA vanishes
- the adjustment U is equal to the *classical bilateral CVA* (see e.g. Gregory [4])
- symmetric prices
- have **justified** issuer hedging own credit **via repurchase of his own bonds**
    - where repurchase is funded through cash account
- but it's difficult in practise to use the derivative as collateral

Case II: $s_F > 0$

If the derivative **cannot** be used as collateral:

- perfect hedging in all scenarios *except* own default
- hedge error always a windfall to the issuer
- windfall cannot be monetised by issuer...
- ...so it appears as the additional FCA cost term in the valuation

Implications

- non-Symmetric derivatives prices so $\hat{V}$ has to be re-interpreted as the issuer production cost
- the market-price will be given by supply and demand
- issuers with lower funding costs than the market clearing level will make excess profits (c.f. the electricity market)
- exact size of the FCA linked to the details of the issuer funding strategy

Balance sheet impact and windfall

Consider simple balance sheet model without the derivative:

- expected recovery: $R_0 = \frac{A_0}{L_0}$
- hazard rate: λ
- funding spread: $(1 - R_0)\lambda$
- funding cost per unit time: $f_0 = (1 - R_0)\lambda L_0$

Now add the derivative d to the balance sheet:

- add asset d and liability d (derivative value un-affected by the default)
- expected recovery after adding d: $R_1 = \frac{A_1}{L_1} = \frac{A_0+d}{L_0+d}$
- new funding cost

$$
\begin{aligned}
f_1 \mathrm{d}t &= (r + (1 - R_1)\lambda) L_1 \mathrm{d}t \\
&= r \cdot d \cdot \mathrm{d}t + f_0 dt
\end{aligned}
$$

Balance sheet impact and windfall (ctd.)

Implications:

- effective cost of funding for derivative is at risk free rate r
- this is because the counterparty has to pay back d in full in cash under ISDA
- hard to make this link in practise
- more direct ways to monetize this windfall and reduce the FCA?
- using derivative as collateral is one (see previous discussion)
  - again, in practise not that straightforward
- managing balance sheet impact is another one
  - possible if one can fund issuing/repurchasing 2 bonds of different recovery

Balance sheet management of funding impact

If issuer can **freely** issue/repurchase own bonds P_1 and P_2 of different recoveries $0 \leq R_1 < R_2 \leq 1$:

$$\begin{cases} \frac{dP_1}{P_1} & = & r_1(t)dt - (1 - R_1)dJ_B \\ \frac{dP_2}{P_2} & = & r_2(t)dt - (1 - R_2)dJ_B \end{cases}$$

Replicating portfolio (S and P_C financed via repo)

$$\begin{aligned} \Pi &= \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \delta S + \beta_S + \beta_C \\ -\hat{V} &= \alpha_1 P_1 + \alpha_2 P_2 \end{aligned}$$

Own default hedged with

$$\begin{aligned} \alpha_1 &= -\frac{R_2 \hat{V} - V^+ - R_B V^-}{(R_2 - R_1)P_1} \\ \alpha_2 &= -\frac{V^+ + R_B V^- - R_1 \hat{V}}{(R_2 - R_1)P_2}, \end{aligned}$$

Balance sheet management of funding impact (ctd.)

Pricing PDE if no basis (i.e. $r_i = r + (1 - R_i)\lambda_B$ for $i = 1, 2$):

$$\partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+$$

Implications:

- same PDE as for $s_F = 0$ in previous section (not surprising since perfect hedging)
- hence $\hat{V} = V + U$ with U being the classical bilateral CVA
- issuing senior P_2-bonds and buying junior P_1-bonds not always possible.
- typically requires posting the derivative as collateral.

$\boxed{\text{What happens if } P_2 \text{ cannot be issued?}}$

A model with one own bond only

If issuer has cannot issue P_2 then $\hat{V} = -\alpha_1 P_1$

- raise all funds by issuing P_1 by necessity
- invest all cash by purchasing P_1 since it offers higher return than P_2
- no degrees of freedom left to try to hedge own default
- hedge out the share and counterparty default with S and P_C as usual

Pricing PDE and decomposition $\hat{V} = V + U$

$$\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r_1 + \lambda_C)\hat{V} = -\lambda_C (R_C V^+ + V^-)$$

$$\begin{aligned} U(t,S) &= -(1-R_C)\int_t^T \lambda_C(u) D_{r_1+\lambda_C}(t,u)\mathbb{E}_t\left[V^+(u,S(u))\right] du \\ &\quad - \int_t^T s_1(u) D_{r_1+\lambda_C}(t,u)\mathbb{E}_t\left[V(u,S(u))\right] du \end{aligned}$$

A model with one own bond only (ctd)

Some model properties:

- setting $\lambda_C = 0$ retrieves the model in Piterbarg [5] for an un-collateralised trade.
  - straightforward to add collateral to the models in this presentation
- the formula for U is different from the earlier setup but has the same structure
- hedge error at own default given by $(1 - R_B)V^+ - R_B U$ (windfall or shortfall)
- the P_1 bond is not able to offset the disappearance of U from the valuation
- no direct reference to the issuer hazard rate λ_B

Summary

Considered hedging strategies for bilateral counterparty risk in the presence of funding costs.

Perfect hedging in all scenarios imply risk-free financing, the classical bilateral CVA and symmetric prices

- but this probably requires the issuer to be able to post the derivative as collateral.

More realistic hedging strategies are imperfect:

- not hedged in scenarios at own default
- impossibility to monetise short/windfall implies additional funding cost terms...
- ...which size depends on the precise funding/hedging strategy used.
- asymmetric prices

Details see Burgard and Kjaer [2], [3].

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