Integrable Boundaries in AdS/CFT

Vidas Regelskis

Department of Mathematics
University of York

ICFT’11
City University London,
April 15, 2011
Non-integrable Singlet Boundary
Outline

Superstrings in AdS/CFT
  The setup
  Worldsheet S-matrix
  Underlying symmetries

Integrable boundaries
  Integrability at the boundary
  D3h/D7h branes and twisted Yangian
  D3v/D7v branes
  D5h/D5v branes and achiral twisted Yangian
Outline

Superstrings in AdS/CFT
   The setup
   Worldsheet S-matrix
   Underlying symmetries

Integrable boundaries
   Integrability at the boundary
   D3h/D7h branes and twisted Yangian
   D3v/D7v branes
   D5h/D5v branes and achiral twisted Yangian
The \textit{AdS/CFT} correspondence, as originally conjectured by J. Maldacena, states an equivalence (or duality) between two very different theories:

- $\mathcal{N} = 4$ super \textit{Yang-Mills} theory in 4d with the gauge group $SU(N)$ and coupling constant $g_{YM}$ in conformal phase;
- Type \textit{IIB superstring} theory on $AdS_5 \times S^5$ where both $AdS_5$ and $S^5$ have the same radius and the coupling constant $g_S$; with the the identification $g_S = g_{YM}^2$.

The \textit{AdS/CFT} conjecture states that these theories, including operator observables, states, correlation functions and full dynamics, are equivalent to each other.
Single Trace Operators

Single trace operators may be represented as spin chains constructed out of excitations \( \chi_i = \{\phi_1, \phi_2, \psi_3, \psi_4\} \) and vacuum reference state \( \mathcal{L} : \)

- **Vacuum reference state**

\[
|0\rangle = |...\mathcal{L} \mathcal{L} ...\mathcal{L} \mathcal{L} ...angle
\]

- **Asymptotic states (excitations)**

\[
|\chi_1...\chi_K\rangle = \sum_{n_1 \ll ... \ll n_K} e^{ip_1n_1} ... e^{ip_Kn_K} |...\mathcal{L} \chi_1 \mathcal{L} ... \mathcal{L} \chi_K \mathcal{L} ...\rangle
\]
Closed Spin Chain

\[ \sum p_i = 2\pi n \]
S-matrix

\[ \frac{\zeta e^{i \phi_1}}{p_1} \quad \frac{\zeta e^{i \phi_1 + i \phi_2}}{p_2} \quad \frac{\zeta e^{i \phi_2}}{p_1} \quad \frac{\zeta e^{i \phi_1 + i \phi_2}}{p_2} \]
The S-matrix is defined as operator acting on the tensor product of vector spaces

\[ S(p_1, p_2) : \quad V(p_1, \zeta) \otimes V(p_2, \zeta e^{ip_1}) \rightarrow V(p_2, \zeta) \otimes V(p_1, \zeta e^{ip_2}). \]

The fundamental S-matrix is fixed by the bosonic symmetries to be of the form

\[
\begin{align*}
S \left| \phi^a \phi^b \right> &= A \left| \phi^a_2 \phi^b_1 \right> + B \left| \phi^a_2 \phi^b_1 \right> + \frac{1}{2} C \varepsilon^{ab} \varepsilon_{\alpha \beta} \left| \psi^\alpha_2 \psi^\beta_1 \right>, \\
S \left| \psi^\alpha \psi^\beta \right> &= D \left| \psi^\alpha_2 \psi^\beta_1 \right> + E \left| \psi^\alpha_2 \psi^\beta_1 \right> + \frac{1}{2} F \varepsilon^{\alpha \beta} \varepsilon_{\alpha \beta} \left| \phi^a_2 \phi^b_1 \right>, \\
S \left| \phi^a \psi^\beta \right> &= G \left| \psi^\beta_2 \phi^a_1 \right> + H \left| \phi^a_2 \psi^\beta_1 \right>, \\
S \left| \psi^\alpha \phi^b \right> &= K \left| \psi^\alpha_2 \phi^b_1 \right> + L \left| \phi^b_2 \psi^\alpha_1 \right>.
\end{align*}
\]
The symmetry algebra of the light-cone superstrings on $\text{AdS}_5 \times S^5$ and of the single trace operators in the $\mathcal{N} = 4$ SYM is $\mathfrak{psu}(2,2|4)$.

The worldsheet $S$-matrix of the theory is manifestly invariant under residual $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \times \mathbb{R}^3$ with common central charges.

\[
\begin{align*}
\{Q_{a}^{\alpha}, Q_{b}^{\beta}\} &= \varepsilon^{ab} \varepsilon_{\alpha\beta} C, \\
\{G_{a}^{\alpha}, G_{b}^{\beta}\} &= \varepsilon^{\alpha\beta} \varepsilon_{ab} C^{\dagger}, \\
\{Q_{a}^{\alpha}, G_{b}^{\beta}\} &= \delta^{a}_{b} L_{\beta}^{\alpha} + \delta_{\beta}^{\alpha} R_{b}^{a} + \delta^{a}_{b} \delta_{\beta}^{\alpha} H,
\end{align*}
\]

here $a, b, c = 1, 2$ and $\alpha, \beta, \gamma = 3, 4$.

[N. Beisert, hep-th/0511082]
Co-products of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$

\[
\Delta R_a^b = R_a^b \otimes 1 + 1 \otimes R_a^b,
\]
\[
\Delta L_\alpha^\beta = L_\alpha^\beta \otimes 1 + 1 \otimes L_\alpha^\beta,
\]
\[
\Delta Q_\alpha^a = Q_\alpha^a \otimes 1 + 1 \otimes Q_\alpha^a,
\]
\[
\Delta G_a^\alpha = G_a^\alpha \otimes 1 + 1 \otimes G_a^\alpha,
\]
\[
\Delta C = C \otimes 1 + 1 \otimes C,
\]
\[
\Delta C^\dagger = C^\dagger \otimes 1 + 1 \otimes C^\dagger,
\]
\[
\Delta H = H \otimes 1 + 1 \otimes H.
\]
Yangian algebra

The Yangian $Y(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a deformation of the universal enveloping algebra $U(\mathfrak{g}[u])$ of the polynomial algebra $\mathfrak{g}[u] = \mathfrak{g} \oplus u\mathfrak{g} \oplus u^2\mathfrak{g} \oplus \ldots$, $[u^r\mathfrak{g}, u^s\mathfrak{g}] \subseteq u^{r+s}\mathfrak{g}$.

It is generated by grade-0 $\mathfrak{g}$ generators $\mathcal{J}^A$ and grade-1 $Y(\mathfrak{g})$ generators $\hat{\mathcal{J}}^A$. Their commutators have the generic form:

$$[\mathcal{J}^A, \mathcal{J}^B] = f^{AB}_C \mathcal{J}^C,$$

$$[\mathcal{J}^A, \hat{\mathcal{J}}^B] = f^{AB}_C \hat{\mathcal{J}}^C,$$

and must obey Jacobi and Serre relations

$$[[\mathcal{J}^A, [\mathcal{J}^B, \mathcal{J}^C]]] = 0,$$

$$[[\mathcal{J}^A, [\hat{\mathcal{J}}^B, \hat{\mathcal{J}}^C]]] = 0,$$

$$[[\hat{\mathcal{J}}^A, [\hat{\mathcal{J}}^B, \mathcal{J}^C]]] = \frac{1}{4} f^{AG}_D f^{BH}_E f^{CK}_F f^{GHK}_{\{DJEKF\}} \mathcal{J}^D \mathcal{J}^E \mathcal{J}^F.$$

The co-products of the grade-0 and grade-1 Yangian generators are:

$$\Delta \mathcal{J}^A = \mathcal{J}^A \otimes 1 + 1 \otimes \mathcal{J}^A,$$

$$\Delta \hat{\mathcal{J}}^A = \hat{\mathcal{J}}^A \otimes 1 + 1 \otimes \hat{\mathcal{J}}^A + \frac{1}{2} f^{A}_{BC} \mathcal{J}^B \otimes \mathcal{J}^C.$$
\[
\Delta \hat{R}_a^b = \hat{R}_a^b \otimes 1 + 1 \otimes \hat{R}_a^b + \frac{1}{2} \hat{R}_a^c \otimes R_c^b - \frac{1}{2} R_c^b \otimes \hat{R}_a^c - \frac{1}{2} G_a^\gamma \otimes Q_\gamma^b - \frac{1}{2} Q_\gamma^b \otimes G_a^\gamma \\
+ \frac{1}{4} \delta_a^b G_c^\gamma \otimes Q_\gamma^c + \frac{1}{4} \delta_a^b Q_\gamma^c \otimes G_c^\gamma,
\]

\[
\Delta \hat{L}_\alpha^\beta = \hat{L}_\alpha^\beta \otimes 1 + 1 \otimes \hat{L}_\alpha^\beta - \frac{1}{2} L_\alpha^\gamma \otimes L_\gamma^\beta + \frac{1}{2} L_\gamma^\beta \otimes L_\alpha^\gamma + \frac{1}{2} G_c^\beta \otimes Q_\alpha^c + \frac{1}{2} Q_\alpha^c \otimes G_c^\beta \\
- \frac{1}{4} \delta_\alpha^\beta G_c^\gamma \otimes Q_\gamma^c - \frac{1}{4} \delta_\alpha^\beta Q_\gamma^c \otimes G_c^\gamma,
\]

\[
\Delta \hat{Q}_\alpha^a = \hat{Q}_\alpha^a \otimes 1 + 1 \otimes \hat{Q}_\alpha^a + \frac{1}{2} Q_\alpha^c \otimes R_c^a - \frac{1}{2} R_c^a \otimes Q_\alpha^c + \frac{1}{2} Q_\gamma^a \otimes L_\alpha^\gamma - \frac{1}{2} L_\alpha^\gamma \otimes Q_\gamma^a \\
+ \frac{1}{4} Q_\alpha^a \otimes H - \frac{1}{4} H \otimes Q_\alpha^a + \frac{1}{2} \varepsilon_\alpha^\gamma \varepsilon^{ac} G_c^\gamma \otimes C + \frac{1}{2} \varepsilon_\alpha^\gamma \varepsilon^{ac} G_c^\gamma \otimes C,
\]

\[
\Delta \hat{G}_a^\alpha = \hat{G}_a^\alpha \otimes 1 + 1 \otimes \hat{G}_a^\alpha - \frac{1}{2} G_c^\alpha \otimes R_c^\alpha + \frac{1}{2} R_c^\alpha \otimes G_c^\alpha - \frac{1}{2} G_a^\gamma \otimes L_\alpha^\gamma + \frac{1}{2} L_\alpha^\gamma \otimes G_a^\gamma \\
- \frac{1}{4} G_a^\alpha \otimes H + \frac{1}{4} H \otimes G_a^\alpha - \frac{1}{2} \varepsilon_{ac} \varepsilon^{\alpha \gamma C^\dagger} \otimes Q_\gamma^c + \frac{1}{2} \varepsilon_{ac} \varepsilon^{\alpha \gamma Q_\gamma^c} \otimes C^\dagger,
\]

\[
\Delta \hat{C} = \hat{C} \otimes 1 + 1 \otimes \hat{C} - \frac{1}{2} H \otimes C + \frac{1}{2} C \otimes H,
\]

\[
\Delta \hat{C}^\dagger = \hat{C}^\dagger \otimes 1 + 1 \otimes \hat{C}^\dagger + \frac{1}{2} H \otimes C^\dagger - \frac{1}{2} C^\dagger \otimes H,
\]

\[
\Delta \hat{H} = \hat{H} \otimes 1 + 1 \otimes \hat{H} + C \otimes C^\dagger - C^\dagger \otimes C.
\]
The $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ supercharges act on this basis states as:

\[ Q^b_\beta |\phi_a\rangle = a \delta^b_a |\psi_\beta\rangle, \quad \mathbb{G}^\beta_b |\phi_a\rangle = c \varepsilon^\beta \varepsilon^{b a} |\psi_\alpha\rangle, \]
\[ Q^b_\beta |\psi_\alpha\rangle = b \varepsilon^{b a} \varepsilon_\beta |\phi_a\rangle, \quad \mathbb{G}^\beta_b |\psi_\alpha\rangle = d \delta^\beta_\alpha |\phi_b\rangle. \]

The representations labels $a, b, c, d$ are

\[ a = \sqrt{\frac{g}{2l}} \eta, \quad b = \sqrt{\frac{g}{2l}} \frac{i \zeta}{\eta} \left( \frac{x^+}{x^-} - 1 \right), \quad c = -\sqrt{\frac{g}{2l}} \frac{\eta}{\zeta x^+}, \quad d = -\sqrt{\frac{g}{2l}} \frac{x^+}{i \eta} \left( \frac{x^-}{x^+} - 1 \right), \]

where $\zeta$ is an overall phase factor, $\eta$ reflects the freedom of the choice of spectral parameters $x^\pm$ obeying

\[ e^{i \rho} = \frac{x^+}{x^-}, \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i \frac{2l}{g}. \]

The invariance of the S-matrix under the symmetry algebra

\[ [\Delta \mathbb{J}^A, S(p_1, p_2)] = 0, \quad \Delta \mathbb{J}^A = \mathbb{J}^A \otimes 1 + 1 \otimes \mathbb{J}^A, \]

constrains all fundamental S-matrix coefficients uniquely up to an overall phase!
S-matrix Coefficients

\[ A = 1, \quad B = 2 \frac{x_2^+ (x_1^- - x_2^-) (-1 + x_1^+ x_2^-)}{x_2^- (x_1^- - x_2^+)} \left( -1 + x_1^+ x_2^+ \right) - 1, \]

\[ C = \frac{i (x_1^- - x_2^-) \tilde{\eta}_1 \tilde{\eta}_2}{\zeta (x_1^- - x_2^+)(-1 + x_1^+ x_2^+)} , \]

\[ D = -\frac{(x_2^- - x_1^+) \tilde{\eta}_1 \tilde{\eta}_2}{(x_1^- - x_2^+) \eta_1 \eta_2} , \]

\[ E = \left( \frac{x_2^- - x_1^+}{x_1^- - x_2^+} + 2 \frac{x_1^+ (x_1^- - x_2^-) (-1 + x_1^- x_2^+)}{x_1^- (x_1^- - x_2^+)(-1 + x_1^+ x_2^+)} \right) \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2} , \]

\[ F = -\frac{i \zeta x_1^+ (x_1^- - x_2^-) (x_1^- - x_1^+) (x_2^- - x_2^+)} {x_1^- x_2^- (x_1^- - x_2^+) (-1 + x_1^+ x_2^+) \eta_1 \eta_2} , \]

\[ G = \frac{(x_1^+ - x_2^+) \tilde{\eta}_2}{(x_1^- - x_2^+) \eta_1} , \quad H = \frac{(x_1^- - x_2^-) \tilde{\eta}_1}{(x_1^- - x_2^+) \eta_1} , \]

\[ K = \frac{(x_1^- - x_1^+) \tilde{\eta}_2}{(x_1^- - x_2^+) \eta_1} , \quad L = \frac{(x_2^- - x_2^+) \tilde{\eta}_1}{(x_1^- - x_2^+) \eta_2} . \]
## Bound-state $S$-matrix

<table>
<thead>
<tr>
<th>Group</th>
<th>Representation</th>
<th>$4 \otimes 4$</th>
<th>$S^{AA}$</th>
<th>$S^{AB}$</th>
<th>$S^{BB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{su}(2</td>
<td>2)$</td>
<td>$\begin{array}{c} \Box \otimes \Box = \Box \oplus \Box \ 4 \otimes 4 = 8 \oplus 8 \end{array}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathfrak{psu}(2</td>
<td>2)_c$</td>
<td>$\begin{array}{c} \Box \otimes \Box = \text{Long} \ 4 \otimes 4 = 16 \end{array}$</td>
<td>$S^{AA}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\begin{array}{c} \Box \otimes \Box = \text{Long} \ 4 \otimes 8 = 32 \end{array}$</td>
<td>$S^{AB}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\begin{array}{c} \Box \otimes \Box = \text{Long} \oplus \text{Long} \ 8 \otimes 8 = 16 \oplus 48 \end{array}$</td>
<td>$S^{BB}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[N. Beisert, M. de Leeuw, A. Torrielli, G. Arutyunov and S. Frolov]
Yang-Baxter Equation (YBE)

\[ S_{23} S_{13} S_{12} = S_{12} S_{13} S_{23} \]
Quick summary of the scattering theory

- The fundamental asymptotic states \( \chi^i = \{ \phi^1, \phi^2, \psi^3, \psi^4 \} \) are functions of momentum \( p \) and phase \( \zeta \) and live in \( \mathbb{R}_{(a,b,c,d)} \).

- The \( S \)-matrix is defined as operator acting on the tensor product of vector spaces:

\[
S(p_1, p_2) : V(p_1, \zeta) \otimes V(p_2, \zeta e^{ip_1}) \rightarrow V(p_2, \zeta) \otimes V(p_1, \zeta e^{ip_2}).
\]

- The matrix elements of the \( S \)-matrix

\[
\langle \chi^k(p_2, \zeta_1) \chi^l(p_1, \zeta_1 e^{ip_2}) | S | \chi^i(p_1, \zeta_1) \chi^j(p_2, \zeta_1 e^{ip_1}) \rangle = a_{ij}^{kl},
\]

are constrained by the underlying symmetries

\[
\langle \chi^k(p_2, \zeta_1) \chi^l(p_1, \zeta_1 e^{ip_2}) | [S, \Delta] | \chi^i(p_1, \zeta_1) \chi^j(p_2, \zeta_1 e^{ip_1}) \rangle = 0,
\]

up to an overall dressing factor.
Outline

Superstrings in AdS/CFT
  The setup
  Worldsheet S-matrix
  Underlying symmetries

Integrable boundaries
  Integrability at the boundary
  D3h/D7h branes and twisted Yangian
  D3v/D7v branes
  D5h/D5v branes and achiral twisted Yangian
Boundary Yang-Baxter Equation (bYBE)

\[ K_2 S_{21} K_1 S_{12} = S_{21} K_1 S_{12} K_2 \]
Integrable Boundaries

For the boundary to be integrable it must respect a (sub)algebra of the bulk algebra \( \mathfrak{h} \subseteq \mathfrak{g} \) and saturate a BPS bound.

The boundaries we shall consider are:

1. D3 ‘Giant graviton’ brane wrapping maximal \( S^3 \subset AdS_5 \times S^5 \)
   1.1 \( Z = 0 \) ‘Giant graviton’ [D.Hofman & J.Maldacena, C.Ahn & D.Bak & S.Rey, N.MacKay & V.R.]
   1.2 \( Y = 0 \) ‘Giant graviton’ [D.Hofman & J.Maldacena, C.Ahn & R.Nepomechie, N.MacKay & V.R., L.Palla]

2. D7 brane wrapping entire \( AdS_5 \) and maximal \( S^3 \subset S^5 \)
   2.1 \( Z = 0 \) D7 brane [D.Correa & C.Young, N.MacKay & V.R.]
   2.2 \( X = 0 \) D7 brane [D.Correa & C.Young, N.MacKay & V.R.]

3. D5 brane wrapping \( AdS_4 \times S^2 \)
   3.1 ‘Vertical’ D5 [D.Correa & C.Young & V.R.]
   3.2 ‘Horizontal’ D5 [D.Correa & C.Young & V.R.]
The bulk algebra is $\mathfrak{psu}(2|2) \times \overline{\mathfrak{psu}}(2|2) \times \mathbb{R}^3$.

<table>
<thead>
<tr>
<th>Brane</th>
<th>Boundary algebra</th>
<th>Boundary rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D3h/D7h$</td>
<td>$\mathfrak{psu}(2</td>
<td>1) \times \overline{\mathfrak{psu}}(2</td>
</tr>
<tr>
<td>$D3v$</td>
<td>$\mathfrak{psu}(2</td>
<td>2) \times \overline{\mathfrak{psu}}(2</td>
</tr>
<tr>
<td>$D7v$</td>
<td>$\mathfrak{su}(2) \times \mathfrak{su}(2) \times \overline{\mathfrak{psu}}(2</td>
<td>2) \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5h$</td>
<td>$\mathfrak{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5v$</td>
<td>$\mathfrak{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
</tbody>
</table>
The bulk algebra is $\text{psu}(2|2) \times \widetilde{\text{psu}}(2|2) \times \mathbb{R}^3$.

<table>
<thead>
<tr>
<th>Brane</th>
<th>Boundary algebra</th>
<th>Boundary rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D3h/D7h$</td>
<td>$\text{psu}(2</td>
<td>1) \times \widetilde{\text{psu}}(2</td>
</tr>
<tr>
<td>$D3v$</td>
<td>$\text{psu}(2</td>
<td>2) \times \widetilde{\text{psu}}(2</td>
</tr>
<tr>
<td>$D7v$</td>
<td>$\text{su}(2) \times \text{su}(2) \times \widetilde{\text{psu}}(2</td>
<td>2) \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5h$</td>
<td>$\text{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5v$</td>
<td>$\text{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
</tbody>
</table>
$K(p): \quad V(p, \zeta) \otimes V_B(0) \rightarrow V(-p, \zeta) \otimes V_B(0)$
Fundamental K-matrix (D3h/D7h)

The $K$-matrix for the reflection of bulk magnons from the boundary vacuum state is defined as:

$$K : \quad V(p, \zeta) \otimes V_B(0) \rightarrow V(-p, \zeta) \otimes V_B(0),$$

Boundary algebra is $\mathfrak{su}(2|1) = \{ L^{\beta}_\alpha, R^1_1, R^2_2, Q^1_\alpha, G^\alpha_1, H \}$. All boundary algebra generators annihilate boundary vacuum state.

The fundamental $K$-matrix may be represented as

$$K |\phi^1_p\rangle = A |\phi^1_{-p}\rangle, \quad K |\phi^2_p\rangle = B |\phi^2_{-p}\rangle, \quad K |\psi^\alpha_p\rangle = C |\psi^\alpha_{-p}\rangle.$$

The invariance of the $K$-matrix under the symmetry algebra

$$(J^A \otimes 1)K(p) - K(p)(J^A \otimes 1) = 0,$$

constrains all $K$-matrix coefficients uniquely up to an overall phase!
**K-matrix (D3h/D7h)**

<table>
<thead>
<tr>
<th></th>
<th>Fundamental states</th>
<th>Two-magnon bound-states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{psu}(2</td>
<td>2) \times \mathbb{R}^3$</td>
<td>$\square \otimes 1_B$</td>
</tr>
<tr>
<td>$\text{su}(2</td>
<td>1)$</td>
<td>$\square \otimes 1_B$</td>
</tr>
</tbody>
</table>
Twisted Yangian of $Y=0$ Giant graviton (D3h/D7h)

Let the boundary algebra be a subalgebra of bulk algebra $\mathfrak{h} \subset \mathfrak{g}$ such that the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ respects the symmetric pair property

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h},$$

This is crucial in guaranteeing the co-ideal property

$$\Delta \hat{\mathfrak{j}} \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h}).$$

We introduce a graded involution $\sigma$ of $\mathfrak{g}$ with the eigenspaces $\sigma(\mathfrak{h}) = +1$ and $\sigma(\mathfrak{m}) = -1$. Then boundary Yangian $Y(\mathfrak{g}, \mathfrak{h})$ may be thought of as a deformation of the subalgebra of $\mathcal{U}(\mathfrak{g}[u])$, which is invariant under the extension $\bar{\sigma}$ of $\sigma$ which sends $\bar{\sigma} : u \mapsto -u$

$$\mathfrak{h} \oplus \mathfrak{m} \oplus \ldots \subset \mathfrak{g}[u] = (\mathfrak{h} \oplus \mathfrak{m}) \oplus u(\mathfrak{h} \oplus \mathfrak{m}) \oplus \ldots$$

Hence the boundary Yangian charges must live in the subspace $\mathfrak{um}$. 
Twisted Yangian of $Y=0$ Giant graviton ($D3h/D7h$)

<table>
<thead>
<tr>
<th></th>
<th>$\bar{\sigma}$</th>
<th>$\bar{\eta}$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>level-0</td>
<td>+</td>
<td>−</td>
<td></td>
</tr>
<tr>
<td>level-1</td>
<td>−</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>level-2</td>
<td>+</td>
<td>−</td>
<td></td>
</tr>
</tbody>
</table>
Twisted Charges

However, while the grade-0 generators of $\mathfrak{h}$ clearly respect the co-ideal property, the grade-1 generators of $\text{um}$ do not do so,

$$\Delta \hat{J}^p = \hat{J}^p \otimes 1 + 1 \otimes \hat{J}^p + \frac{1}{2} f_{qi}^p \left( \hat{J}^q \otimes \hat{J}^i + \hat{J}^i \otimes \hat{J}^q \right) \notin Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h}),$$

where $i, j$ run over the $\mathfrak{h}$-indices and $p, q$ over the $\mathfrak{m}$-indices.

Rather we need a deformation of the grade-1 $\text{um}$ generators, and therefore we find $Y(\mathfrak{g}, \mathfrak{h})$ to be the algebra generated by $\{ \hat{J}^i, \tilde{\hat{J}}^p \}$, where

$$\tilde{\hat{J}}^p := \hat{J}^p + \frac{1}{2} f_{qi}^p \hat{J}^q \hat{J}^i,$$

are the twisted boundary Yangian generators.
Twisted Charges

However, while the grade-0 generators of $\mathfrak{h}$ clearly respect the co-ideal property, the grade-1 generators of $\mathfrak{um}$ do not do so,

$$\Delta \hat{J}^p = \hat{J}^p \otimes 1 + 1 \otimes \hat{J}^p + \frac{1}{2} f^p_{qi} \left( \hat{J}^q \otimes \hat{J}^i + \hat{J}^i \otimes \hat{J}^q \right) \notin Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h}),$$

where $i, j$ run over the $\mathfrak{h}$-indices and $p, q$ over the $\mathfrak{m}$-indices.

Rather we need a deformation of the grade-1 $\mathfrak{um}$ generators, and therefore we find $Y(\mathfrak{g}, \mathfrak{h})$ to be the algebra generated by $\{ \hat{J}^i, \tilde{\hat{J}}^p \}$, where

$$\tilde{\hat{J}}^p := \hat{J}^p + \frac{1}{2} f^p_{qi} \hat{J}^q \hat{J}^i,$$

are the twisted boundary Yangian generators.
Twisted Charges

Now we can show that $\mathcal{Y}(g, \mathfrak{h})$ is a left co-ideal subalgebra, $\Delta \mathcal{Y}(g, \mathfrak{h}) \subset \mathcal{Y}(g) \otimes \mathcal{Y}(g, \mathfrak{h})$. To do this we calculate explicitly the co-product of the twisted Yangian generator:

$$\Delta \tilde{\mathcal{J}}^p = \Delta \hat{\mathcal{J}}^p + \frac{1}{2} f^p_{qi} \Delta J^q \Delta J^i$$

$$= \hat{\mathcal{J}}^p \otimes 1 + 1 \otimes \hat{\mathcal{J}}^p + \frac{1}{2} f^p_{qi} \left( J^q J^i \otimes 1 + 1 \otimes J^q J^i \right)$$

$$+ \frac{1}{2} f^p_{iq} \hat{\mathcal{J}}^i \otimes J^q + \frac{1}{2} f^p_{qi} J^q \otimes J^i + \frac{1}{2} f^p_{qi} \left( J^q \otimes J^i + J^i \otimes J^q \right)$$

$$= \tilde{\mathcal{J}}^p \otimes 1 + 1 \otimes \tilde{\mathcal{J}}^p + f^p_{qi} J^q \otimes J^i$$

$$\in \mathcal{Y}(g) \otimes \mathcal{Y}(g, \mathfrak{h}),$$

where we have used the symmetric pair property - the only non-zero structure constants in this case are $f^p_{qi}$ and $f^p_{iq}$. 
Twisted Yangian $Y(\mathfrak{psu}(2|2)_C, \mathfrak{su}(2|1))$

\[
\Delta \tilde{R}_1^2 = \left( \hat{R}_1^2 + \frac{1}{2} R_1^2 R_1^1 - \frac{1}{2} R_1^2 R_2^2 - \frac{1}{2} Q_\gamma^2 G_1^\gamma \right) \otimes 1,
\]

\[
\Delta \tilde{R}_2^1 = \left( \hat{R}_2^1 + \frac{1}{2} R_2^1 R_1^1 - \frac{1}{2} R_2^1 R_2^2 - \frac{1}{2} G_2^\gamma Q_\gamma^1 \right) \otimes 1,
\]

\[
\Delta \tilde{Q}_\alpha^2 = \left( \hat{Q}_\alpha^2 + \frac{1}{2} Q_\alpha^2 R_2^2 - \frac{1}{2} R_1^2 Q_\alpha^1 + \frac{1}{2} Q_\gamma^2 L_\alpha^\gamma + \frac{1}{4} Q_\alpha^2 H - \frac{1}{2} \epsilon_\alpha^\gamma C G_1^\gamma \right) \otimes 1,
\]

\[
\Delta \tilde{G}_2^\alpha = \left( \hat{G}_2^\alpha - \frac{1}{2} G_2^\alpha R_2^2 + \frac{1}{2} R_2^1 G_1^\alpha - \frac{1}{2} G_2^\gamma L_\gamma^\alpha - \frac{1}{4} G_2^\alpha H + \frac{1}{2} \epsilon_\alpha^\gamma C^\dagger Q_\gamma^1 \right) \otimes 1,
\]

\[
\Delta \tilde{C} = \left( \hat{C} + \frac{1}{2} C H \right) \otimes 1,
\]

\[
\Delta \tilde{C}^\dagger = \left( \hat{C}^\dagger - \frac{1}{2} C^\dagger H \right) \otimes 1.
\]

The bulk algebra is $\text{psu}(2|2) \times \tilde{\text{psu}}(2|2) \times \mathbb{R}^3$.

<table>
<thead>
<tr>
<th>Brane</th>
<th>Boundary algebra</th>
<th>Boundary rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D3h/D7h$</td>
<td>$\text{psu}(2</td>
<td>1) \times \tilde{\text{psu}}(2</td>
</tr>
<tr>
<td>$D3v$</td>
<td>$\text{psu}(2</td>
<td>2) \times \tilde{\text{psu}}(2</td>
</tr>
<tr>
<td>$D7v$</td>
<td>$\text{su}(2) \times \text{su}(2) \times \tilde{\text{psu}}(2</td>
<td>2) \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5h$</td>
<td>$\text{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5v$</td>
<td>$\text{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
</tbody>
</table>
\[ K(p, q) : \quad V(p, \zeta) \otimes V_B(q, \zeta e^{ip}) \rightarrow V(-p, \zeta) \otimes V_B(q, \zeta e^{-ip}) \]
The $K$-matrix for the reflection of bulk magnons from the boundary states is defined as

$$K(p,q) : \quad V(p,\zeta) \otimes V_B(q,\zeta e^{ip}) \to V(-p,\zeta) \otimes V_B(q,\zeta e^{-ip}),$$

and is fixed by the bosonic symmetries to be of the form

$$K \left| \phi_p^a \phi_q^b \rightangle = A \left| \phi_{-p}^{\{a} \phi_q^{b\}} \rightangle + B \left| \phi_{-p}^{[a} \phi_q^{b]} \rightangle + \frac{1}{2} C \epsilon^{ab} \epsilon_{\alpha\beta} \left| \psi_{-p}^\alpha \psi_q^\beta \rightangle,$$

$$K \left| \psi_p^\alpha \psi_q^\beta \rightangle = D \left| \psi_{-p}^{\{\alpha} \psi_q^{\beta\}} \rightangle + E \left| \psi_{-p}^{[\alpha} \psi_q^{\beta]} \rightangle + \frac{1}{2} F \epsilon^{\alpha\beta} \epsilon_{ab} \left| \phi_{-p}^a \phi_q^b \rightangle,$$

$$K \left| \phi_p^a \psi_q^b \rightangle = G \left| \psi_{-p}^\beta \phi_q^a \rightangle + H \left| \phi_{-p}^a \psi_q^\beta \rightangle,$$

$$K \left| \psi_p^\alpha \phi_q^b \rightangle = K \left| \psi_{-p}^\alpha \phi_q^b \rightangle + L \left| \phi_{-p}^b \psi_q^\alpha \rightangle.$$

The invariance of the $K$-matrix under the symmetry algebra

$$\left[ \Delta \mathbb{J}^A, K(p,q) \right] = 0, \quad \Delta \mathbb{J}^A = \mathbb{J}^A \otimes 1 + 1 \otimes \mathbb{J}^A,$$

constrains all fundamental $K$-matrix coefficients uniquely up to an overall phase!
<table>
<thead>
<tr>
<th>$\square \otimes \square_B$</th>
<th>$\text{Long}$</th>
<th>$4 \otimes 4 = 16$</th>
<th>$K^{Aa}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\square \otimes \square_B$</td>
<td>$\text{Long}$</td>
<td>$4 \otimes 8 = 32$</td>
<td>$K^{Ab}$</td>
</tr>
<tr>
<td>$\square \otimes \square_B$</td>
<td>$\text{Long}$</td>
<td>$8 \otimes 4 = 32$</td>
<td>$K^{Ba}$</td>
</tr>
<tr>
<td>$\square \otimes \square_B$</td>
<td>$\text{Long} \oplus \text{Long}$</td>
<td>$8 \otimes 8 = 16 \oplus 48$</td>
<td>$K^{Bb}$</td>
</tr>
</tbody>
</table>
The bulk algebra is $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2) \times \mathbb{R}^3$.

<table>
<thead>
<tr>
<th>Brane</th>
<th>Boundary algebra</th>
<th>Boundary rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D3h/D7h$</td>
<td>$\mathfrak{psu}(2</td>
<td>1) \times \widetilde{\mathfrak{psu}}(2</td>
</tr>
<tr>
<td>$D3v$</td>
<td>$\mathfrak{psu}(2</td>
<td>2) \times \widetilde{\mathfrak{psu}}(2</td>
</tr>
<tr>
<td>$D7v$</td>
<td>$\mathfrak{su}(2) \times \mathfrak{su}(2) \times \widetilde{\mathfrak{psu}}(2</td>
<td>2) \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5h$</td>
<td>$\mathfrak{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5v$</td>
<td>$\mathfrak{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
</tbody>
</table>
### K-matrix \((D7v_L)\)

<table>
<thead>
<tr>
<th></th>
<th>Fundamental</th>
<th>Bound-states</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{psu}(2</td>
<td>2) \times \mathbb{R}^3)</td>
<td>(\square \otimes 1_B)</td>
</tr>
<tr>
<td>(\text{su}(2)_L \times \text{su}(2)_R \times \mathbb{R})</td>
<td>((\square_L \oplus \square_R) \otimes 1_B)</td>
<td>((\square_R \oplus \square_L \oplus \square_{L/R}) \otimes 1_B)</td>
</tr>
</tbody>
</table>
The bulk algebra is $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2) \times \mathbb{R}^3$.

<table>
<thead>
<tr>
<th>Brane</th>
<th>Boundary algebra</th>
<th>Boundary rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D3h/D7h$</td>
<td>$\mathfrak{psu}(2</td>
<td>1) \times \widetilde{\mathfrak{psu}}(2</td>
</tr>
<tr>
<td>$D3v$</td>
<td>$\mathfrak{psu}(2</td>
<td>2) \times \widetilde{\mathfrak{psu}}(2</td>
</tr>
<tr>
<td>$D7v$</td>
<td>$\mathfrak{su}(2) \times \mathfrak{su}(2) \times \widetilde{\mathfrak{psu}}(2</td>
<td>2) \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5h$</td>
<td>$\mathfrak{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$D5v$</td>
<td>$\mathfrak{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
</tr>
</tbody>
</table>
### K-matrix (D5h/D5v)

<table>
<thead>
<tr>
<th></th>
<th>D5h</th>
<th>D5v</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{psu}(2</td>
<td>2) \times \text{psu}(2</td>
<td>2) \times \mathbb{R}^3$</td>
</tr>
<tr>
<td>$\text{psu}(2</td>
<td>2)_+ \times \mathbb{R}^3$</td>
<td>$\begin{array}{c} \square_+ \circ \square_+ \times 1_B \end{array}$</td>
</tr>
</tbody>
</table>
Achiral twisted Yangian of D5h/D5v branes

The algebra in the bulk is $\text{psu}(2|2) \times \widetilde{\text{psu}}(2|2) \times \mathbb{R}^3 \equiv g_L \oplus g_R$. The boundary algebra is $\text{psu}(2|2)_+ \times \mathbb{R}^3 \equiv g_+$. We write the symmetric pair structure as $g_L \oplus g_R = g_+ \oplus g_-$. These are spanned by $J_\pm^a = J_L^a \pm \alpha(J_R^a)$.

Then the boundary Yangian symmetry $Y(g_L \times g_R, g_+)$ is generated by the level-0 gens $J_+^a$ and twisted level-1 gens $\tilde{J}_-^a$

\[
\tilde{J}_-^a := \hat{J}_-^a + \frac{1}{8} f_{ab}^c (J_-^c J_+^b + J_+^b J_-^c)
\]

\[
= \hat{J}_-^a + \frac{1}{2} f_{bc}^a J_L^b \alpha(J_R^c),
\]
The co-product of which is

\[
\Delta \tilde{j}_a = \Delta \hat{j}_a + \frac{1}{8} f^a_{cb} (\Delta j^b_+ \Delta j^c_- + \Delta j^c_- \Delta j^b_+ ) \\
= \hat{j}_a \otimes 1 + 1 \otimes \hat{j}_a + \frac{1}{8} f^a_{cb} (j^b_+ j^c_- + j^c_- j^b_+) \otimes 1 + \frac{1}{8} f^a_{cb} 1 \otimes (j^b_+ j^c_- + j^c_- j^b_+) \\
+ \frac{1}{4} f^{ab}_{bc} (j^b_- \otimes j^c_+ + j^b_+ \otimes j^c_- ) + \frac{1}{4} f^a_{cb} (j^b_- \otimes j^c_+ - j^b_+ \otimes j^c_- ) \\
= \tilde{j}_a \otimes 1 + 1 \otimes \tilde{j}_a + \frac{1}{2} f^a_{bc} j^b_- \otimes j^c_+ \\
\in Y(g_L \times g_R) \otimes Y(g_L \times g_R, g_+) 
\]
Achiral twisted Yangian of D5h/D5v branes

The co-product of twisted $Y(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ charge may be written as

$$\Delta \tilde{J}_-^a = \left( \hat{J}_L^a \circ 1 - 1 \circ \alpha \left( \hat{J}_R^a \right) + \frac{1}{2} f_{bc}^a \hat{J}_L^b \circ \alpha \left( \hat{J}_R^c \right) \right) \otimes 1$$

$$+ 1 \otimes \left( \hat{J}_-^a + \frac{1}{8} f_{cb}^a \left( \hat{J}_-^c \hat{J}_+^b + \hat{J}_+^b \hat{J}_-^c \right) \right)$$

$$+ \left( \frac{1}{2} f_{bc}^a \left( \hat{J}_L^b \circ 1 - 1 \circ \alpha \left( \hat{J}_R^b \right) \right) \right) \otimes \hat{J}_+^c.$$

The co-product of any $Y(\mathfrak{g})$ charge is

$$\Delta \hat{J}^a = \hat{J}^a \otimes 1 + 1 \otimes \hat{J}^a + \frac{1}{2} f_{bc}^a \hat{J}^b \otimes \hat{J}^c.$$
D5h brane

The reflection matrix is a map

\[ K^h : \square \circ \tilde{\square} \otimes 1 \to \square \circ \tilde{\square} \otimes 1, \]

and may be neatly represented on superspace as an operator

\[ K^h : V(p, \zeta) \circ V(-p, \zeta e^{ip}) \rightarrow V(-p, \zeta) \circ V(p, \zeta e^{-ip}) \]

The non-trivial part of the co-product of twisted \( Y(g \times g, g) \) charge is

\[ \Delta \hat{\mathcal{J}}^a = \left( \hat{\mathcal{J}}_L^a \circ 1 - 1 \circ \alpha \left( \hat{\mathcal{J}}_R^a \right) + \frac{1}{2} f_{bc}^a \hat{\mathcal{J}}_L^b \circ \alpha \left( \hat{\mathcal{J}}_R^c \right) \right) \otimes 1 \]

and effectively differs from the the co-product of \( Y(g) \) charge

\[ \Delta \hat{\mathcal{J}}^a = \hat{\mathcal{J}}^a \otimes 1 + 1 \otimes \hat{\mathcal{J}}^a + \frac{1}{2} f_{bc}^a \hat{\mathcal{J}}^b \otimes \hat{\mathcal{J}}^c \]

by the minus signs only!
D5h brane: folded and unfolded pictures of the reflection

\[
(-u, -p, \zeta) \sim (-u, p, \zeta e^{-ip})
\]

\[
S(p, -p) \quad \kappa \quad S(p, -p)
\]

\[
(u, p, \zeta) \sim (u, -p, \zeta e^{ip})
\]

\[
(u, p, \zeta) \quad \kappa \quad (u, p, \zeta e^{-ip})
\]

\[
(-u, -p, \zeta) \quad \kappa \quad (-u, -p, \zeta e^{ip})
\]
D5v brane

The reflection matrix is a map
\[ K^v : \Box \circ \tilde{\Box} \otimes \tilde{\Box} \to \Box \circ \tilde{\Box} \otimes \tilde{\Box}, \]
which factorizes as a composition of a bulk S-matrix and two achiral reflection matrices \( \kappa \)
\[ K^v(p, p) = \kappa(p, x_B) S(p, -p) \kappa(p, x_B). \]

The non-trivial part of the co-product of twisted \( Y(g \times g, g) \) charge is
\[ \Delta \tilde{J}^a_\pm = \left( \hat{J}^a_L \circ 1 - 1 \circ \alpha \left( \hat{J}^a_R \right) + \frac{1}{2} f^a_{bc} \hat{J}^b_L \circ \alpha \left( \hat{J}^c_R \right) \right) \otimes 1 \]
\[ + \left( \frac{1}{2} f^a_{bc} \left( \hat{J}^b_L \circ 1 - 1 \circ \alpha \left( \hat{J}^b_R \right) \right) \right) \otimes \hat{J}^c_+, \]
and is effectively composed of S-matrix
\[ \Delta \tilde{J}^a_\pm \bigg|_S = \left( \hat{J}^a_L \circ 1 - 1 \circ \alpha \left( \hat{J}^a_R \right) + \frac{1}{2} f^a_{bc} \hat{J}^b_L \circ \alpha \left( \hat{J}^c_R \right) \right) \otimes 1, \]
and \( \kappa \)-matrix
\[ \Delta \tilde{J}^a_\pm \bigg|_\kappa = \left( \hat{J}^a_L \circ 1 - 1 \circ \alpha \left( \hat{J}^a_R \right) \right) \otimes 1 + \frac{1}{2} f^a_{bc} \left( \hat{J}^b_L \circ 1 - 1 \circ \alpha \left( \hat{J}^b_R \right) \right) \otimes \hat{J}^c_+ \]
parts.
D5v brane: folded and unfolded pictures of the reflection

\[ (-u, -p, \zeta) (-u, -p, \zeta) (x_B, \zeta e^{-ip}) \]

\[ S(p, -p) \]

\[ \kappa(p, x_B) \]

\[ (u, p, \zeta) (u, p, \zeta) (x_B, \zeta e^{ip}) \]

\[ \simeq \]

\[ (-u, -p, \zeta) (x_B, \zeta e^{-ip}) (u, -p, \zeta) \]

\[ S(p, -p) \]

\[ \kappa(p, x_B) \]

\[ (u, p, \zeta) (x_B, \zeta e^{ip}) (-u, p, \zeta) \]
D5v brane: LLM-type disc diagram
Folding-unfolding
Summary

- We have constructed fundamental and bound-state K-matrices (\(\kappa\)-matrices) for various configurations of D3, D7 and D5 branes and showed that these boundaries are integrable.

- We have constructed of twisted Yangian \(Y(g, h)\) and achiral twisted Yangian \(Y(g \times g, g)\) that govern the reflection from D3h/D7h and D5h/D5v branes respectively.

- We showed that achiral reflection is closely related to the scattering in the bulk.

Outlook

- We need to understand what are the underlying Yangian structures for D3v/D7v branes
- What is the unifying picture?