Department of Economics
School of Social Sciences

An Asset Pricing Model for Mean-Variance-Downside-Risk Averse Investors

Jose Olmo[1]
City University

Department of Economics
Discussion Paper Series
No. 07/01

[1] Department of Economics, City University, Northampton Square, London, EC1V 0HB, UK. Email:j.olmo@city.ac.uk

An Asset Pricing Model for Mean-Variance-Downside-Risk Averse Investors

Jose Olmo, *City University, London*[*]

This draft, January 2007

Abstract

We introduce a family of utility functions that describe the preferences of mean-variance-downside-risk (mvdr) averse investors. The risk premium on a risky asset in an economy with these individuals is given by a weighted sum of CAPM systematic risk and a systematic risk given by the level of comovements between the asset and the market in distress episodes. Hence investors require a higher reward than predicted by CAPM for holding assets correlated with the market in distress episodes, and a lower reward for holding assets with negative correlation in market downturns. The application of this pricing theory to financial sectors in FTSE-100 is illuminating. The empirical failure of standard CAPM is explained by the extra reward required by investors from market downturns. While Chemicals and Mining sectors exhibit positive comovements with FTSE downturns; Banking and Oil and Gas sectors are robust to them and Telecommunications Services exhibit negative comovements serving as refugee of investors fleeing from domestic market distress episodes.

JEL code: G11, G12, G13.

Keywords: Asset Pricing, CAPM, Downside-risk, Mean-variance

[*]Corresponding Address: Dept. Economics, City University. Northampton Square, EC1 V0HB, London. Jose Olmo, E-mail:j.olmo@city.ac.uk.

1 Introduction

The Nobel laureate William Sharpe showed in 1964 that under equilibrium conditions the price of a risky asset in an economy of mean-variance agents is determined by the covariance between the returns on the risky asset and on the market portfolio. This model extended the outstanding work of another recipient of the Nobel prize, Harry Markowitz who in 1952 formalized the concept of diversification for investors' optimal asset allocation problem. These models found wide support within academics and practitioners by its simplicity and tractability.

The empirical implementation of CAPM however was fraught with difficulties from an early stage. The careful work of Black, Jensen and Scholes (1972) demonstrates that assets with low β earn a higher return on average than predicted by CAPM and high β assets earn a lower return. Fama and French (1992) found that the univariate relation between β and average return on risky assets for the 1941-1990 period is weak. In other words, β does not suffice to explain the average return on a risky investment. These authors also explored the use of other variables to explain the variation in average returns. In particular Fama and French (1995) find that firm's size and book-to-market equity do a good job at explaining the cross-section of average returns on financial equity markets for the 1963-1990 period.

These theories shed some doubts on CAPM formulations; thereby some authors started to postulate other models considering downside-risk measures. Markowitz (1959) proposed the semivariance, that was later refined by Hogan and Warren (1974). Bawa (1975) and Bawa and Lindenberg (1977) propose minimizing Lower Partial Moments (LPM) of the distribution of returns as alternative to the variance for constructing optimal portfolios. Building on this theory Harlow and Rao (1989) introduce the generalized Mean-Lower Partial Moment (MLPM) model that they use for asset pricing. This theory generalizes earlier pricing formulations enclosing static CAPM or LPM models as particular cases.

The empirical evidence on different downside-risk models at explaining shortcomings of CAPM is mixed however. Jahankhani (1976) concludes that mean-semivariance models do not perform any better than standard mean-variance CAPM. In the same line Harlow and Rao (1989) argue that downside-risk formulations using the risk-free return as threshold are not successful in describing the risk premium of a risky investment. These authors claim that the relevant benchmark target seems to be implied by the data. It is the mean of the distribution of returns rather than the risk-free rate.

The aim of this paper is to stress the significance of both models. Variance and downside-risk are the factors driving the price of a risky asset in an economy of risk averse investors. However CAPM fails as a pricing model solely relying on the covariance between returns; and downside-risk CAPM is not successful at forecasting the risk premium on a risky investment

by gauging simply the dependence of its yield with market returns on downturns periods.

In this paper we devise an economy of mean-variance-downside-risk averse investors with preferences modelled by a family of utility functions penalizing separately departures from expected wealth and wealth levels below a threshold. This family of functions build on the spirit of those introduced in Bawa (1975), (1976), (1978) and Bawa and Lindenberg (1977). With this family the subsequent investors' optimal portfolio decision problem and asset pricing model are natural extensions of the above mentioned CAPM and downside-risk models. The risk premium in our model is proportional to the market portfolio as in standard formulations. In contrast to previous models however, the systematic risk is given by a weighted sum of a systematic risk coming from the overall dependence of the asset with the market and from a systematic risk given by the level of comovements between the asset and the market in distress episodes.

The results in this paper are consistent with those of Ang, Chen and Xing (2001), (2005) using a downside-risk asset pricing model and Post and Vliet (2004) using a mean-semivariance model. These authors show there is a premium for holding stocks with a higher downside risk. Stocks that are highly correlated with the market when the market declines have higher expected returns than stocks that are not so correlated. By turning attention to London Stock Exchange we also find evidence of two related results: stocks negatively correlated with the market during market downturns have lower premium than the rest of stocks because serve as refugee for investors, and CAPM holds for robust sectors to market downturns.

The paper is structured as follows. Section 2 introduces the formulation of the model. This consists of a version of the family of utility functions introduced in Bawa (1975) and Fishburn (1977) but incorporating both variance and different measures of downside-risk. This section also examines the subsequent market model and pricing equilibrium formulas. The statistical representation of the economic model is developed in Section 3. The next section studies different financial sectors trading in the London Stock Exchange (LSE) and comprising *FTSE*-100. Finally Section 5 summarizes the main contributions of the paper. Mathematical derivations and tables are gathered in the appendices.

2 Asset Pricing Theories

Markowitz (1952) showed that investors should select assets as if they only care about the mean and variance of returns. The outstanding contribution of this theory is that the risk underlying an investment decision can be measured by one simple statistical measure: the variance. This risk measure is consistent with the maximization of the expected value of utility functions of quadratic form or exhibiting constant absolute risk aversion (CARA). Examples of these

families of utility functions are

$$u(W) = W - bW^2, \quad with \quad b > 0, \tag{1}$$

or

$$u(W) = 1 - \exp^{(-\phi W)}, \quad with \quad \phi > 0. \tag{2}$$

We will use W to denote investors' wealth and R defined by $R = (W - W_0)/W_0$ to denote the simple return on a W_0 investment. The distribution of returns is assumed to be normal for CARA investors.

Sharpe (1964) and Litner (1965) extended investor's decision problem under uncertainty to an economy of mean-variance agents. The capital asset pricing model (CAPM) shares the virtues of Markowitz's theory in what is simple and tractable. The risk premium required by investors for holding risky assets is proportional to the risk premium of a market portfolio. The static CAPM formula reads as

$$E[R_j] - R_f = \beta_j(E[R_m] - R_f) \tag{3}$$

with R_j and R_m risky returns on an asset j and on a market portfolio, R_f is the risk-free return and β_j a constant. The other virtue inherited from Markowitz's model is that

$$\beta_j = \frac{\sigma_{jm}}{\sigma_m^2}, \tag{4}$$

with σ_{jm} standing for the covariance between the risky return and the return on the market portfolio, and σ_m^2 standing for the market portfolio variance.

This theory has been exhaustively revised and refined in the literature. Fama and French (1992), (1993), (1995), Black, Jensen and Scholes (1972) or Merton (1973) are some remarkable examples. The striking conclusion of this asset pricing theory is that variance and covariance guide the trade-off between risk and return. Risk in these economies is identified with statistical measures gauging uncertainty and linear dependence.

Downside-risk averse investors on the other hand have a different perception of risk. This is identified with the likelihood of a dread event. Bawa (1975) and Arzac and Bawa (1977) elaborate on this idea of risk and define risk measures based on lower partial moments (LPM) of the distribution of returns. Bawa (1975), (1976), (1978) develops a class of utility functions for downside-risk averse investors consistent with these alternative risk measures. It takes this

form

$$u(R;n,\tau) = \begin{cases} a + bR - c(\tau - R)^n, & \text{for} \quad R \leq \tau, \\ a + bR, & \text{for} \quad R > \tau, \end{cases} \tag{5}$$

where a, b, and c are constants, and τ denotes a target return.

Bawa and Lindenberg (1977) and Harlow and Rao (1989) extend investors' optimal portfolio choice problem given by the following formula

$$\min_X LPM_n(\tau; X) = \int_{-\infty}^{\tau} (\tau - X'R)^n dF(R), \tag{6}$$

subject to $X_0 R_f + \sum_j X_j E[R_j] = \mu$ and $X_0 + \sum_j X_j = 1$, to an economy driven by downside-risk averse investors.

In equilibrium conditions the risk premium on a risky asset j is given by

$$E[R_j] - R_f = \beta_j^{mlpm_n(\tau)} (E[R_m] - R_f) \tag{7}$$

with

$$\beta_j^{mlpm_n(\tau)} = \frac{\int_{-\infty}^{\tau} \int_{-\infty}^{\infty} (\tau - R_m)^{n-1} (R_f - R_j) dF(R_j, R_m)}{\int_{-\infty}^{\infty} (\tau - R_m)^{n-1} (R_f - R_j) dF(R_m)}. \tag{8}$$

A risky asset traded in both economies will have the same price for $n = 2$, $\tau = R_f$ and if the distribution of returns is normal.

The striking conclusion of this model is that downside-risk averse investors are not concerned about the variance of returns. The price of an asset in equilibrium is given by the covariance of the asset with the market in turmoil periods. This raises several challenges for asset pricing; the choice of the threshold must be the same for each individual in the economy; if the economy has not gone through any distress episode there is no way in practice of pricing the asset; and finally bull equity markets driving the price of the market portfolio up have no effect on the price of the risky asset.

If we confine ourselves to $n = 2$, LPM risk measures for a variable R are of this form

$$LPM_2(\tau) = \Big(V[R|R \leq \tau] + (E[R|R \leq \tau] - \tau)^2 \Big) F(\tau). \tag{9}$$

The proof of this result is immediately derived from adding and substracting $E[R|R \leq \tau]$ into the integrand in (6).

Investors' decisions are only influenced by $E[R|R \leq \tau]$ and $V[R|R \leq \tau]$ with $\{R \leq \tau\}$ a subset of the domain of R denoting dread events. Thus, two random variables with same shortfall probability and conditional expected value below τ have equal risk even if their variances are different. In this context upper partial moments of R are not taken into account

for measuring risk. However the uncertainty about the outcome, provided it is positive, also entails certain risk that is missmeasured with these formulations. Unknown excess returns, even if positive, hinder future investment plans. In a macroeconomic policy context for example the aim of achieving future inflation values close to a target inflation penalizes departures in both directions.

To illustrate this result we introduce the following example. Let X, Y be two random variables sharing the following distributional moments;

$$\begin{aligned} P\{X \leq \tau\} &= P\{Y \leq \tau\}, \\ E[X] &= E[Y], \\ E[X|X \leq \tau] &= E[Y|Y \leq \tau], \\ V[X|X \leq \tau] &= V[Y|Y \leq \tau], \end{aligned}$$

but with the variance of X greater than the variance of Y.

By the law of iterated expectations for the variance,

$$V[X] = E[V[X|A]] + V[E[X|A]] \tag{10}$$

with A denoting different conditioning sets covering the domain of R. In this example,

$$V[X] = V[X|X \leq \tau]P\{X \leq \tau\} + V[X|X > \tau]P\{X > \tau\} + (E[X|X \leq \tau] - E[X])^2 P\{X \leq \tau\} + (E[X|X > \tau] - E[X])^2 P\{X > \tau\}.$$

By the law of iterated expectations for the mean,

$$E[X] = E[X|X \leq \tau]P\{X \leq \tau\} + E[X|X > \tau]P\{X > \tau\}. \tag{11}$$

The same decomposition applies to Y. Then it is immediate to see that

$$E[X|X > \tau] = E[Y|Y > \tau]$$

and in turn

$$V[X|X > \tau] > V[Y|Y > \tau]. \tag{12}$$

The mean-variance-downside-risk (*mvdr*) economy postulated in this paper is a compromise between the two worlds. By considering the variance investors penalize the uncertainty underlying exceedances of the target. By considering downside-risk separately investors are allowed to punish observations below the target in a different fashion than deviations beyond it. We propose the following family of utility functions to describe the preferences of

mean-variance-downside-risk averse individuals.

$$u(W) = aW - bW^2 - c(\tau - W)^n I(W \leq \tau), \tag{13}$$

with a, b, and c constants, and $I(W \leq \tau)$ an indicator function taking a value of 1 if W is less than the target τ and 0 otherwise. In particular we will study the case $n = 1$,

$$u(W) = \begin{cases} (a+c)W - bW^2 - c\tau & if \quad W \leq \tau, \\ aW - bW^2 & if \quad W > \tau. \end{cases} \tag{14}$$

If an investor has a portfolio P and its preferences are described by a utility function of type (14) the optimal portfolio allocation will result from maximizing $E[u(W_p)]$ with W_p denoting terminal wealth on portfolio P. The fundamental valuation relationships (derived from the first order conditions of the maximization problem) for this case yield the following equation (see Appendix A for the algebra)

$$E[R_j] - R_f = \beta_j^{mvdr}(E[R_p] - R_f) \tag{15}$$

with

$$\beta_j^{mvdr} = \left(1 - \frac{Cov(R_p, I(R_p \leq \tau^*))}{\gamma\sigma_p^2 + Cov(R_p, I(R_p \leq \tau^*))}\right)\beta_j + \frac{\sigma_I^2}{\gamma\sigma_p^2 + Cov(R_p, I(R_p \leq \tau^*))}\beta_j^{\tau^*}, \tag{16}$$

where $\tau^* = \frac{\tau}{W_0} - 1$ is a transformation of the original threshold on wealth, $\sigma_I = \sigma_{I(R_p \leq \tau^*)}$, $\gamma = -2bW_0/c$, and $\beta_j^{\tau^*}$ is the dependence parameter implied by the threshold. This takes this form

$$\beta^{\tau^*} = \frac{Cov(R_j, I(R_p \leq \tau^*))}{\sigma_I^2}.$$

The efficient portfolio frontier for *mvdr* averse investors is a straight line with slope β_j^{mvdr}. In the following section we extend this method to asset pricing in a *mvdr* economy.

2.1 Mean-variance-downside-risk CAPM: mvdr-CAPM

In the traditional CAPM of Sharpe (1964) and Lintner (1965) the market portfolio is a convex combination of the whole spectrum of risky assets available in the market. All investors in this model are assumed to assign the same probabilities to future investment cash flows and use the same expected returns, variances and correlations. The only difference among their investment strategies stems from their degree of risk aversion that is reflected in the proportion of their portfolio invested in the risk-free asset. These individuals choose to hold efficient portfolios in

the sense that they minimize the variance given the level of expected wealth.

In an economy consisting of downside-risk averse investors an efficient portfolio choice involves an optimal allocation of wealth determined by $\beta^{mlpm(\tau)}$ between the risk-free asset and the market portfolio. As Bawa and Lindenberg (1977) and Harlow and Rao (1989) note the two-fund separation results of Ross (1978) also hold for LPM_n measures for $n = 1, 2$ if the distribution of returns belongs to a two-parameter, location-scale class. In this case standard results in the mean-variance space apply to mean-LPM_n space. The efficiency of the value-weighted market portfolio for mean-variance and downside-risk averse investors is discussed in Post and van Vliet (2006).

We show in the following that in an economy of mean-variance-downside-risk averse investors the risk premium on a risky asset is given by formulas of type (15) and (16) adapted to the market (the market portfolio R_m replaces the optimal portfolio R_p). In order to see this let us assume there are B agents in the economy. These have homogeneous beliefs and construct the same objective probabilities for the distribution of returns. We also assume that every agent in the economy has the same threshold τ^* on the return on the investment. Note that this assumption does not preclude economies of individuals with different levels of risk aversion represented by τ_i, $i = 1, \ldots, B$.

Market equilibrium implies clearing of the assets traded in the market. Thus, asset prices adjust such that the proportion of each asset invested in every investor's risky portfolio equals the share of that asset in the market portfolio. Let $i = 1, \ldots, B$ index the investors in the market. The total value of asset j in the market is $p_j X_j$ where X_j denotes the total amount invested in that risky asset. This value corresponds to aggregate every investor value in asset j, *i.e.*, $p_j X_j = \sum_{i=1}^{B} p_j x_{ji}$ with x_{ji} denoting investor's i amount in asset j. The corresponding share in security j of investor's i holding of all risky assets (W_i) is $z_{ji} = \frac{p_j x_{ji}}{W_i}$. Therefore

$$p_j X_j = \sum_{i=1}^{B} z_{ji} W_i.$$

If investors in this economy are only distinguishable by their budget constraint and their threshold level on wealth their share of investment in each risky asset will be identical. Note the optimal share in each asset is determined by formulas (15) and (16) and these equations depend on τ^* that is identical across individuals. Then $z_{j1} = \ldots = z_{jN} = z_j$ with N denoting number of risky assets in the economy. Substituting in the previous equality we obtain

$$p_j X_j = z_j W,$$

with W denoting total wealth in the economy and defined by the sum of individuals' wealth.

It immediately follows that $z_j = \frac{p_j X_j}{W}$ with z_j the share of asset j in the market portfolio M. This portfolio M plays the role of portfolio P for each investor's optimization problem.

Consider a portfolio Q of q assets traded in the market, with $q \leq N+1$. The return on portfolio Q is defined as $R_q = \sum_{j=1}^{q} w_j R_j$ where w_j denotes the proportion in asset j and satisfies $\sum_{j=1}^{q} w_j = 1$. The risk premium required by investors on Q is given by this formula,

$$E[R_q] - R_f = \beta_q^{mvdr} \left(E[R_m] - R_f \right), \tag{17}$$

with

$$\beta_q^{mvdr} = \left(1 - \frac{Cov(R_m, I(R_m \leq \tau^*))}{\gamma \sigma_m^2 + Cov(R_m, I(R_m \leq \tau^*))} \right) \beta_q + \frac{\sigma_I^2}{\gamma \sigma_m^2 + Cov(R_m, I(R_m \leq \tau^*))} \beta_q^{\tau^*}, \tag{18}$$

and where $\beta_q = \sum_{j=1}^{q} w_j \beta_j$ and $\beta_q^{\tau^*} = \sum_{j=1}^{q} w_j \beta_j^{\tau^*}$. Note that σ_I^2 in this context denotes the variance of $I(R_m \leq \tau^*)$.

The risk premium of a risky asset j traded in an economy of *mvdr* averse investors is proportional to the risk premium on the market portfolio as in CAPM or downside-risk CAPM. It differs from the risk premium on these pricing models in what the systematic risk depends both on the covariance and the level of comovements with the market portfolio. Comovements is defined as conditional covariance between R_j and R_m under market downturns.

3 Econometric Model

The usual econometric model for testing CAPM implications is

$$R_{jt} = \alpha + \beta_j R_{mt} + \eta_{jt}, \tag{19}$$

with η_{jt} denoting an *iid* random variable following a normal distribution with zero mean. Though this model measures cross-sectional dependence it is estimated with time series data from realized excess returns of both portfolios.

The empirical representations of downside risk models share the spirit of model (19). In particular Harlow and Rao (1989) modify the asymmetric response model proposed by Bawa, Brown and Klein (1981) in testing the Bawa-Lindenberg model to obtain a statistical representation of the $MLPM_n$ model. This is

$$R_{jt} = \alpha_j + \beta_{1j} R_{mt}^- + \beta_{2j} R_{mt}^+ + \varphi(\beta_{1j} - \beta_{2j}) I(R_{mt} > \tau) + \eta_{jt}, \quad j = 1, \ldots, N; t = 1, \ldots, T, \tag{20}$$

where $R_{mt}^{-} = R_m$ if $R_{mt} < \tau$ and zero otherwise, and $R_{mt}^{+} = R_m$ if $R_{mt} > \tau$ and zero otherwise. The parameter φ denotes the conditional expected excess return determined by τ. Mathematically this reads $\varphi = E[R_{mt}|R_{mt} > \tau]$.

We build on this econometric model to represent the equilibrium pricing relationship found in (17) and (18). Rearranging β^{mvdr} for asset j we obtain

$$\beta_j^{mvdr} = (1-\nu)\,\beta_j + \frac{\sigma_I^2}{Cov(R_m, I(R_m \leq \tau^*))}\nu\beta_j^{\tau^*}, \tag{21}$$

with $\nu = \frac{Cov(R_m, I(R_m \leq \tau^*))}{\gamma\sigma_m^2 + Cov(R_m, I(R_m \leq \tau^*))}$.

Further, after some algebra it can be seen (Appendix B) that

$$\beta_j^{mvdr} = (1-\nu)\,\beta_j + \frac{\nu}{E[R_m|R_m \leq \tau^*]}\beta_j^{\tau^*}. \tag{22}$$

To simplify the model and make it econometrically tractable we further assume $\nu = 1/2$. This boils down to take $\gamma = \frac{Cov(R_m, I(R_m \leq \tau^*))}{\sigma_m^2}$. In terms of the utility function in (14) this assumption constraints the number of free parameters describing investors' preferences. Constants b and c are linked by $c = \frac{-2bW_0\sigma_m^2}{Cov(R_m, I(R_m \leq \tau^*))}$.

Equation (17) for an asset j is equal to

$$E[R_j] = R_f + \frac{1}{2}\beta_j E[R_m] + \frac{1/2}{\phi}\beta_j^{\tau^*}\phi E[I(R_m \leq \tau^*)] - \beta_j^{mvdr}R_f,$$

with $\phi = E[R_{mt}|R_{mt} \leq \tau^*]$. The statistical representation of this model is the following

$$R_{jt} = \widetilde{\alpha}_j + \widetilde{\beta}_j R_{mt} + \widetilde{\beta}_j^{\tau^*}\phi I(R_{mt} \leq \tau^*) + \eta_{jt}, \quad j = 1, \ldots, N; t = 1, \ldots, T, \tag{23}$$

with parameters given by $\widetilde{\alpha}_j = \left(1 - \widetilde{\beta}_j - \widetilde{\beta}_j^{\tau^*}\right) R_f$, $\widetilde{\beta}_j = (1/2)\beta_j$ and $\widetilde{\beta}_j^{\tau^*} = \frac{1/2}{\phi}\beta_j^{\tau^*}$.

This is equivalent to

$$2R_{jt} = 2\widetilde{\alpha}_j + \beta_j R_{mt} + \beta_j^{\tau^*} I(R_{mt} \leq \tau^*) + \eta_{jt}, \quad j = 1, \ldots, N; t = 1, \ldots, T. \tag{24}$$

Ordinary least squares and maximum likelihood are efficient and consistent estimators of the parameters in this model. This representation nests standard and downside-risk CAPM and allows to test statistically the significance of the covariance and comovements between assets in asset pricing. This can be carried out by simple Wald and likelihood ratio tests.

4 Application: London Stock Exchange

Data in this section consist of weekly prices of some of the major stocks traded in London Stock Exchange (LSE) and are obtained from Yahoo-Finance. In order to minimize nuisance factors derived from idiosyncratic risks of each stock we consider sectoral indices. The values analyzed are Aerospace and Defense, Banks, Chemicals, Mining, Oil and Gas, Telecommunications Services and Transport. The market portfolio R_m is proxied by FTSE-100. Both sectoral indices and market portfolio are market-weighted-valued portfolios. The data collected cover the period November 2003-April 2006 (150 observations). The time series published by Bank of England corresponding to Three-month treasury bills is entertained to compute the return on a risk-free asset. These returns are weekly compounded to obtain weekly observations ($R_{f,w} = R_{f,m}/4$) with $R_{f,w}$ weekly return and $R_{f,m}$ monthly return on a Three-month treasury bill.

The section entertains three different pricing models: standard CAPM, a simple downside-risk CAPM, and *mvdr*-CAPM postulated in this paper. The threshold τ^* assumed for models two and three is the risk-free asset. The estimates of the regression equations from models one and two are reported in table 5.1.

(INSERT TABLE (5.1))

Results in table (5.1) highlight two facts: *a)* both intercepts are statistically significant, *b)* comovements in the tails help to explain the risk premium on a risky asset. The significance of α_C is an stylized fact usually found in empirical implementations of CAPM. Black, Jensen and Scholes (1972) state that *assets with low β earn a higher return on average than predicted by CAPM and high β assets earn a lower return,* and Fama and French (2003) point in the same direction by saying that *the intercepts in time series regressions of excess asset returns on the excess market return are positive for assets with low betas and negative for assets with high betas.* Therefore this pricing model fails to solely describe the factors influencing risky asset pricing. These authors partially solve this problem by adding size of firm and book-to-equity ratio as explanatory variables in the standard CAPM formulation (Fama and French, 1995).

We instead extend Sharpe and Lintner formulation by adding downside-risk concerns into the model. In practice however this is not feasible for the variable describing market downturns in model (24) is found statistically not significant though significant in a simple regression model. This fact points towards multicollinearity problems given by a strong correlation

between explanatory variables. This is

$$Corr(R_m, I(R_m \leq \tau^*)) = \frac{E[R_m|R_m \leq \tau^*] - E[R_m]P\{R_m \leq \tau^*\}}{\sqrt{V[R_m]P\{R_m \leq \tau^*\}(1 - P\{R_m \leq \tau^*\})}}. \tag{25}$$

In our example the correlation between FTSE-index returns and the corresponding indicator function is -0.79. To overcome this technical problem we use $I(R_{aerospace} \leq \tau^*)$ to proxy $I(R_{ftse} \leq \tau^*)$. This variable exhibits a moderately high correlation (0.68) with the tail of the return on the market portfolio defined by τ^*, but a low correlation (-0.35) with the unconstrained variable R_{ftse}. The choice of this variable is somehow ad-hoc. Aerospace is selected because is a sector tracking *FTSE* in distress episodes but without much influence in its overall performance. Other candidates playing the same role are Mining and Chemicals, very influential in the performance of the market portfolio in crises episodes but not in its overall performance.

The econometric testable equation is the following:

$$2R_j = 2\widetilde{\alpha}_j + \beta_j R_{ftse} + \beta_j^{R_f} I(R_{aerospace} \leq R_f) + \eta_j \tag{26}$$

with η_j the error term.

Table (5.2) reports estimates from this model for different economic sectors in *LSE*.

(INSERT TABLE (5.2))

Results drawn from table 5.2 are illuminating. The intercept in this model is of opposite sign and similar magnitude than the downside risk parameter. The statistical significance of α_C found in table (5.1) is now compensated by the significance of the tail parameter β^{R_f}. Investors expect a higher reward than predicted by CAPM for assets highly correlated with the market in crises episodes. These are Aerospace and Defense, Chemicals, Mining and Transport sector. Banks, and Oil and Gas sectors are indifferent to market downturns; investors do not price this type of risk when investing in these assets. These sectors are viewed as safe assets. Finally, the Telecommunications sector exhibits negative comovements with the market portfolio. Investors require a lower risk premium than predicted by CAPM because this sector performs well under turmoil market periods. Similar results are found using $I(R_{chemicals} \leq \tau^*)$ or $I(R_{mining} \leq \tau^*)$ as proxies for tracking *FTSE* downturns. The regression equations using these explanatory variables are available from the author upon request.

5 Conclusions

Mean-variance asset pricing does not take into account the presence of comovements between a risky asset and the market portfolio in market downturns unless returns are normally distributed. Downside-risk averse investors on the other hand are only concerned about this risk and do not put a price to the relation between risky returns in bull markets. We overcome the inconsistencies of each standpoint in asset pricing by developing an economy of mean-variance-downside-risk averse investors. Individuals in this economy price assets proportional to the risk premium on the market portfolio but in contrast to preceding asset pricing models the reward for holding the asset is given by measures of dependence between the assets in calm and distress episodes of the market portfolio.

We explain the stylized facts not described by static CAPM model by introducing an extra term rewarding investors for the presence of comovements. In contrast to CAPM, the intercept in the pricing model can be positive to compensate negative returns on the risky asset produced by market downturns. By the same token, other assets can exhibit a negative intercept for they have negative comovements with the market portfolio. Investors require a lower compensation for holding these assets because they provide coverage from market distress episodes. We find empirical evidence of these phenomena from the analysis of stocks of major sectors trading in London Stock Exchange. Banks and Oil and Gas sectors are not influenced by downturns of *FTSE*-100; Chemicals and Mining sectors are correlated with the market in crises episodes and their risk premia reflect this. Finally, the Telecommunications sector serves as refugee from turmoil periods for the risk premium required from investing in this asset is lower than predicted by standard *CAPM*.

References

[2] Ang, A., J.S. Chen, and Y. Xing (2001). "Downside Risk and the Momentum Effect." NBER Working Paper No. W8643. Available at SSRN: http://ssrn.com/abstract=294082

[2] Ang, A., J.S. Chen, and Y. Xing (2005). "Downside Risk." Forthcoming Review of Financial Studies.

[3] Arzac, E., and V. Bawa (1977). "Portfolio choice and equilibrium in capital markets with safety first investors." Journal of Financial Economics 4, 277-288.

[4] Bawa, V.S. (1975). "Optimal rules for ordering uncertain prospects." Journal of Financial Economics 2, 95-121.

[5] Bawa, V.S. (1976). "Admissible portfolios for all individuals." Journal of Finance 23, 1169-1183.

[6] Bawa, V.S. (1978). "Safety first, stochastic dominance, and optimal portfolio cChoice." Journal of Financial and Quantitative Analysis 13, 255-271.

[7] Bawa, V.S., and E. Lindenberg (1977). "Capital market equilibrium in a Mean, Lower Partial Moment Framework." Journal of Financial Economics 5, 189-200.

[8] Bawa, V.S., S. Brown, and R. Klein (1981). "Asymmetric Response Asset Pricing Models: Testable Alternatives to Mean-Variance." Mimeo.

[9] Black ,F., M.C. Jensen, and M. Scholes (1972). "The Capital Asset Pricing Model: some empirical tests." In M. Jensen, ed.: Studies in the Theory of Capital Markets (Praeger).

[12] Fama, E., F., and K. R. French (1992). "The cross-section of expected stock returns." Journal of Finance 47, 427-465.

[12] Fama, E., F., and K. R. French (1993). "Common risk factors in the returns on stock and bonds stock returns." Journal of Financial Economics 33, 3-56.

[12] Fama, E., F., and K. R. French (1995). "Size and Book-to-Market factors in Earnings and Returns." Journal of Finance 50, 1, 131-155.

[13] Fama, E., F., and K. R. French (2003). "The Capital Asset Pricing Theory: Theory and Evidence." CRSP Working Paper No. 550 ; Tuck Business School Working Paper No. 03-26. Available at SSRN: http://ssrn.com/abstract=440920 or DOI: 10.2139/ssrn.440920

[14] Fishburn, P. (1977). Mean-Risk Analysis with Risk Associated with Below-Target returns." American Economic Review 67, 116-126.

[15] Harlow, W.V., and R.K.S. Rao (1989). "Asset Pricing in a Generalized Mean-Lower Partial Moment Framework: Theory and Evidence," Journal of Financial and Quantitative Analysis 3, 285-309.

[16] Hogan, W., and J. Warren (1974). "Toward the development of an Equilibrium Capital-Market Model based on Semivariance." Journal of Financial and Quantitative Analysis 9, 1-11.

[17] Jahankhani, R. (1976). "E-V and E-S capital asset pricing models: some empirical tests." Journal of Financial and Quantitative Analysis 11, 513-528.

[18] Lintner, J. (1965). "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets." Review of Economic and Statistics 47,13-37.

[20] Markowitz, H. (1952). "Portfolio selection." Journal of Finance 7, 77-91.

[20] Markowitz, H. (1959). "Portfolio selection, Efficient Diversification of Investments," Cowles Foundation for Research in Economics at Yale University, Monograph 16, John Wiley & Sons, New York.

[21] Merton, E.C. (1973). "An intertemporal capital asset pricing model." Econometrica 41, 867-887.

[22] Post, T., and P. van Vliet (2004). "Conditional Downside Risk and the CAPM." ERIM Report Series Reference No. $ERS - 2004 - 048 - F\&A$

[23] Post, T., and P. van Vliet (2006). "Downside risk and asset pricing." Journal of Banking and Finance 30, 823-849.

[24] Ross, "Mutual Fund Separation in Financial Theory: Separating Distributions," Journal of Economic Theory 17, 254-286.

[25] Sharpe, W. (1964). "Capital asset prices: A theory of market equilibrium under conditions of risk." Journal of finance 19, 425-442.

Appendix A

We will derive the fundamental valuation relationship in the mean-variance-downside-risk model. The utility function describing these investors' preferences is given by

$$u(W) = aW - bW^2 - c(\tau - W)I(W \leq \tau).$$

We will assume investors hold a portfolio P of N risky assets with prices at t, p_j, $j = 1, \ldots, N$, and a risk-free asset with price p_f. We will use x_j to denote the amounts invested in each asset and W_0 to denote initial wealth (budget constraint). Terminal wealth is given by $W_P = v_f x_f + v_{1k} x_1 + \ldots + v_{Nk} x_N$ where v_{jk} is the gain obtained from investing on asset j in scenario k. The return on investment on security j in the kth-scenario is defined by $R_{jk} = \frac{v_{jk}}{W_0} - 1$. v_f denotes a secure gain derived from investing on a risk-free security with return R_f.

The maximization problem is

$$\underset{x_j}{Max} \quad E[u(W_p)]$$

s.t. $p_f x_f + p_1 x_1 + \ldots + p_N x_N \leq W_0$.

This can be written as

$$\underset{x_j}{Max} \quad E[u(W_p)] - \lambda(p_f x_f + p_1 x_1 + \ldots + p_N x_N - W_0)$$

with λ a shadow parameter.

Deriving with respect to each x_j we obtain the first order conditions, denoted fundamental valuation relationships (FVR). For asset 1 the first order condition is the following,

$$u'(W_{p1})(1 + R_{11})p_1 + \ldots + u'(W_{pK})(1 + R_{1K})p_1 - \lambda p_1 = 0 \tag{27}$$

with $u'(W)$ denoting the first derivative of the function $u(W)$. Equation (27) reads then as

$$E[u'(W_p)(1 + R_1)] = \lambda.$$

This condition is identical across risky securities. Then, for security j we have

$$E[u'(W_p)(1 + R_j)] = \lambda.$$

For the risk-free asset this is

$$E[u'(W_p)(1 + R_f)] = \lambda.$$

By using some simple algebra we derive the following equation,

$$E[u'(W_p)(R_j - R_f)] = 0.$$

By the definition of covariance the previous expression can be written as

$$Cov(u'(W_p), R_j) + E[u'(W_p)]\,(E[R_j] - R_f) = 0.$$

This condition is satisfied for any risky asset held by the investor. In particular for portfolio P. Then

$$Cov(u'(W_p), R_p) + E[u'(W_P)]\,(E[R_p] - R_f) = 0.$$

Rearranging the previous two equations we obtain the relationship between risk premiums of portfolio P and any security j in the portfolio. This is

$$E[R_j] - R_f = \frac{Cov(u'(W_p), R_j)}{Cov(u'(W_p), R_p)}\,(E[R_p] - R_f)\,. \tag{28}$$

The next step is to evaluate the covariance between the marginal utility and risky returns. For mean-variance-downside-risk averse investors this is $u'(W_p) = a - 2bW_p + cI(W_p \leq \tau)$. Using that $W_p = (1 + R_p)W_0$ the numerator of (28) is

$$Cov(u'(W_p), R_j) = -2bW_0 Cov(R_p, R_j) + cCov(I(R_p \leq \frac{\tau}{W_0} - 1), R_j).$$

Then

$$E[R_j] - R_f = \frac{-2bW_0 Cov(R_p, R_j) + cCov(I(R_p \leq \frac{\tau}{W_0} - 1), R_j)}{-2bW_0 Cov(R_p, R_p) + cCov(I(R_p \leq \frac{\tau}{W_0} - 1), R_p)}\,(E[R_p] - R_f)\,.$$

If we denote $\gamma = \frac{-2bW_0}{c}$ and $\tau^* = \frac{\tau}{W_0} - 1$ the slope of the straight line in (28) is now

$$\beta_j^{mvdr} = \frac{\sigma_{p,j} + \frac{1}{\gamma}Cov(I(R_p \leq \tau^*), R_j)}{\sigma_p^2 + \frac{1}{\gamma}Cov(I(R_p \leq \tau^*), R_p)}.$$

After some algebra we find that β_j^{mvdr} can be written as

$$\beta_j^{mvdr} = \frac{\sigma_{p,j}}{\sigma_p^2} + \frac{\sigma_p^2 \frac{1}{\gamma}Cov(I(R_p \leq \tau^*), R_j) - \sigma_{p,j}\frac{1}{\gamma}Cov(I(R_p \leq \tau^*), R_p)}{\sigma_p^2\left(\sigma_p^2 + \frac{1}{\gamma}Cov(I(R_p \leq \tau^*), R_p)\right)}.$$

Then

$$\beta_j^{mvdr} = \left(1 - \frac{Cov(R_p, I(R_p \leq \tau^*))}{\gamma\sigma_p^2 + Cov(R_p, I(R_p \leq \tau^*))}\right)\beta_j + \frac{\sigma_I^2}{\gamma\sigma_p^2 + Cov(R_p, I(R_p \leq \tau^*))}\beta_j^{\tau^*}.$$

with $\sigma_I = \sigma_{I(R_p \leq \tau^*)}$, and $\beta_j^{\tau^*}$ defined as

$$\beta_j^{\tau^*} = \frac{Cov(R_j, I(R_p \leq \tau^*))}{\sigma_I^2}.$$

Appendix B

We obtain the exact form of the parameter β^{mvdr} measuring systematic risk in a economy driven by mean-variance-downside-risk averse investors. From the fundamental valuation relationships found in Section 2 we know that

$$\beta_j^{mvdr} = (1-\nu)\,\beta_j + \frac{\sigma_I^2}{Cov(R_m, I(R_m \leq \tau^*))}\nu\beta_j^{\tau^*}, \tag{29}$$

with $\nu = \frac{Cov(R_m, I(R_m \leq \tau^*))}{\gamma\sigma_m^2 + Cov(R_m, I(R_m \leq \tau^*))}$, $\sigma_I = \sigma_{I(R_m \leq \tau^*)}$ and $\beta_j^{\tau^*} = \frac{Cov(R_j, I(R_m \leq \tau^*))}{\sigma_I^2}$.
By definition of covariance of random variables,

$$Cov(R_m, I(R_m \leq \tau^*)) = E[R_m I(R_m \leq \tau^*)] - E[R_m]E[I(R_m \leq \tau^*)].$$

By the law of iterated expectations,

$$E[R_m I(R_m \leq \tau^*)] = E[E[R_m I(R_m \leq \tau^*)|R_m \leq \tau^*]].$$

Then, $E[R_m I(R_m \leq \tau^*)] = E[R_m|R_m \leq \tau^*]E[I(R_m \leq \tau^*)]$ and the covariance reads as

$$Cov(R_m, I(R_m \leq \tau^*)) = (E[R_m|R_m \leq \tau^*] - E[R_m])\,E[I(R_m \leq \tau^*)].$$

The indicator function $I(R_m \leq \tau^*)$ is a bernoulli random variable with parameter $P\{R_m \leq \tau^*\}$. Then $\sigma_I^2 = E[I(R_m \leq \tau^*)](1 - E[I(R_m \leq \tau^*)])$. Hence

$$\frac{\sigma_I^2}{Cov(R_m, I(R_m \leq \tau^*))} = \frac{1 - E[I(R_m \leq \tau^*)]}{E[R_m|R_m \leq \tau^*] - E[R_m]}.$$

Dividing both numerator and denominator by $E[R_m|R_m \leq \tau^*]$ we obtain

$$\frac{\sigma_I^2}{Cov(R_m, I(R_m \leq \tau^*))} = \frac{1}{E[R_m|R_m \leq \tau^*]}$$

given that $E[R_m] = E[R_m|R_m \leq \tau^*]E[I(R_m \leq \tau^*)]$.

Therefore

$$\beta_j^{mvdr} = (1-\nu)\,\beta_j + \frac{\nu}{E[R_m|R_m \leq \tau^*]}\beta_j^{\tau^*}.$$

Appendix C

Model/Sector		AEDE	BANK	CHEM	MIN	OIL	TEL	TRAN
CAPM	α_c	0.28 (0.19)	-0.17^* (0.11)	0.37^* (0.18)	0.97^* (1.64)	0.38^* (1.23)	-0.23 (0.98)	0.82 (0.02)
	β	0.91^* (0.13)	0.93^* (0.07)	1.21^* (0.12)	1.64^* (0.20)	1.23^* (0.13)	1^* (0.15)	0.82^* (0.11)
	R^2	0.27	0.56	0.44	0.35	0.40	0.27	0.32
d-CAPM	α_d	0.86^* (0.35)	0.42^* (0.23)	1.13^* (0.36)	2.07^* (0.54)	1.41^* (0.36)	0.58 (0.39)	0.39 (0.30)
	β^{R_f}	-1.76^* (0.40)	-1.21^* (0.26)	-2.32^* (0.41)	-3.24^* (0.62)	-2.69^* (0.42)	-2.15^* (0.44)	-1.39^* (0.34)
	R^2	0.13	0.27	0.21	0.18	0.25	0.16	0.12

Table 5.1. *Upper panel displays estimates of α_c and β in the standard CAPM regression equation for excess returns. Standard deviations are in brackets. $(*)$ denotes statistical significance at 5% level. Lower panel displays estimates corresponding to a simple downside risk CAPM model: $R_j = \alpha_{dj} + \beta_j^{R_f} I(R_{ftse} \leq R_f) + \eta_j$. R^2 describes the proportion of variability explained by the model. The sectors analyzed are Aerospace and Defense, Banks, Chemicals, Mining, Oil and Gas, Telecommunications Services and Transport.*

Model/Sector		AEDE	BANK	CHEM	MIN	OIL	TEL	TRAN
mvdr-CAPM	α	4.26^* (0.38)	-0.32 (0.30)	1.23^* (0.49)	1.96^* (0.80)	0.26 (0.54)	-1.46^* (0.60)	1.12^* (0.44)
	β	0.92^* (0.20)	1.90^* (0.16)	2.19^* (0.26)	2.92^* (0.42)	2.45^* (0.29)	2.26^* (0.32)	1.46^* (0.23)
	β^{R_f}	-5.53^* (0.48)	0.20 (0.39)	-1.50^* (0.63)	-2.26^* (1.01)	0.007 (0.70)	1.54^* (0.76)	-1.07^* (0.55)
	R^2	0.65	0.56	0.46	0.38	0.40	0.29	0.34

Table 5.2. *Estimates of coefficients in regression equation (26). Standard deviations are in brackets. $(*)$ denotes statistical significance at 5% level. R^2 describes the proportion of variability explained by the model. The sectors analyzed are Aerospace and Defense, Banks, Chemicals, Mining, Oil and Gas, Telecommunications Services and Transport.*