Consistent valuations with bilateral counterparty risk and funding

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15 February 2012
Overview

1. Introduction into bilateral counterparty risk

2. PDE and Feynman-Kac representations of derivatives with bilateral counterparty risk and funding costs

3. Balance sheet impact

4. Summary
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4. Summary
Intro

Early 2000s ...
Intro

Early 2000s ... 2008 ...

Funding

CVA
Intro

Early 2000s ... 2008 ... 2010 ...

Balance Sheet

CVA

Funding
CVA has gone bilateral ...

Counterparty Risk

Bilateral CVA

II

Unilateral CVA

+

DVA

Balance Sheet

Funding
Funding: there are benefits, there are costs ...
Balance sheet has assets and liabilities ... a lot of boxes ...
... but they are related - DVA and Funding Benefits
... and FCA is related to balance sheet.
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4 Summary
Framework: replication

Extend BS PDE to include:
- bilateral counterparty risk
- funding costs.

Set up portfolio:
- \( \hat{V}(S, t, J_B, J_C) \): derivative value (to issuer)
- \( J_B, J_C \): default indicators
- \( S \): underlying stock
- \( P_C \): counterparty risky bond (zero recovery)
- \( P_B \): issuer risky bond (zero recovery)
- \( \beta \): cash accounts

Carefully consider funding of all positions
Derivative value at default

Default of counterparty or issuer:
- Claim based on derivative mark-to-market value $M$.
- ISDA master agreement seems to suggest $M = V$, where $V$ is the risk-less derivative value.
- Then:
  \[
  \hat{V}(t, S, 1, 0) = V^+(t, S) + R_B V^-(t, S) \quad \text{B defaults first}
  \]
  \[
  \hat{V}(t, S, 0, 1) = R_C V^+(t, S) + V^-(t, S) \quad \text{C defaults first}
  \]
- Brigo and Morini [1] discuss alternatives to this assumption.
- Burgard and Kjaer [2] also consider the case $M = \hat{V}$. 
Asset price dynamics:

- For simplicity assume deterministic credit.

\[
\frac{dS}{S} = \mu dt + \sigma dW \\
\frac{dP_C}{P_C} = r_C dt - dJ_C \\
\frac{dP_B}{P_B} = r_B dt - dJ_B
\]

- \( r_B \) and \( r_C \): yields of the risky (zero recovery) bonds.
- Assumptions:
  - asset \( S \) unaffected by defaults of \( B \) or \( C \).
  - two independent Poisson processes \( J_B \) and \( J_C \).
Replication strategy

Replication of risky derivative:

- \( \hat{V}(t) = \Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t) \)
- Hedge out own credit risk: buy back \( \alpha_B \) own bonds
- Hedge out counterparty credit risk: go short \( \alpha_C \) counterparty bonds

Impose self-financing:

- \( -d\hat{V}(t) = \delta(t)dS(t) + \alpha_B(t)dP_B(t) + \alpha_C(t)dP_C(t) + d\beta(t) \)
Decompose change in cash \( d\beta(t) \):

\[
d\beta(t) = d\beta_S(t) + d\beta_C(t) + d\beta_F(t)
\]

Funding of share position:

\[
d\beta_S(t) = \delta(t)(\gamma_S(t) - q_S(t))S(t)dt
\]
- \( \gamma_S \): dividend income
- \( q_S \): net share position financing costs

Funding of counterparty bond position (short):

\[
d\beta_C(t) = -\alpha_C(t)r(t)P_C(t)dt
\]
- short counterparty bond through repo
Remaining cash position after purchase of own bonds:

\[ d\beta_F(t) = r(t)(-\hat{V} - \alpha_B P_B)^+ dt + r_F(t)(-\hat{V} - \alpha_B P_B)^- dt \]

If positive: invest in risk-free assets.
- don’t add own credit risk (we just hedged it)

If negative: need to fund via external funding provider.
- costs \( r_F \).
- \( r_F = r \) if the derivative can be posted as collateral.
- \( r_F = r + s_F \) with \( s_F > 0 \) if the derivative cannot be posted as collateral.
The replication strategy then becomes:

\[
-d\hat{V} = \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\beta_S + d\beta_C + d\beta_F
\]

\[
= \left\{ -r\hat{V} + s_F(-\hat{V} - \alpha_B P_B)^- + (\gamma_S - q_S)\delta S \\
+ (r_B - r)\alpha_B P_B + (r_C - r)\alpha_C P_C \right\} dt
\]

\[-\alpha_B P_B dJ_B - \alpha_C P_C dJ_C + \delta dS.\]
Replication strategy, cont.

On the other hand, by Ito’s lemma:

\[ d\hat{V} = \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2} \sigma^2 S^2 \partial^2_S \hat{V} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C, \]

with

\[ \Delta \hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0), \quad \Delta \hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) \]

so we can eliminate the risk factors by choosing:

\[ \delta = -\partial_S \hat{V}, \]

\[ \alpha_B = \frac{\Delta \hat{V}_B}{P_B} \]

\[ \alpha_C = \frac{\Delta \hat{V}_C}{P_C} \]
Resulting PDE:

\[ \partial_t \hat{V} + A_t \hat{V} - r \hat{V} = s_F (\hat{V} + \Delta \hat{V}_B)^+ - \lambda_B \Delta \hat{V}_B - \lambda_C \Delta \hat{V}_C \]

\[ = s_F V^+ - (R_B \lambda_B + \lambda_C) V^- - (\lambda_B + R_C \lambda_C) V^+ \]

where

\[ A_t V \equiv \frac{1}{2} \sigma^2 S^2 \partial_S^2 V + (q_S - \gamma_S) S \partial_S V \]

\[ s_F \equiv r_F - r \]

\[ \lambda_B \equiv r_B - r \]

\[ \lambda_C \equiv r_C - r \]
## Hedge positions

<table>
<thead>
<tr>
<th>Asset</th>
<th>Pre default</th>
<th>Post Cparty default</th>
<th>Post Issuer default</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$-\partial S \hat{V}$</td>
<td>$-\partial S \hat{V}$</td>
<td>$-\partial S \hat{V}$</td>
</tr>
<tr>
<td>$\beta_S$</td>
<td>$\partial S \hat{V}$</td>
<td>$\partial S \hat{V}$</td>
<td>$\partial S \hat{V}$</td>
</tr>
<tr>
<td>$P_C$</td>
<td>$-(1 - R_C)V^+ - U$</td>
<td>$(1 - R_C)V^+ + U$</td>
<td>$(1 - R_C)V^+ + U$</td>
</tr>
<tr>
<td>$\beta_C$</td>
<td>$(1 - R_C)V^+ + U$</td>
<td>$-(1 - R_B)V^- - U$</td>
<td>$-(1 - R_B)V^- - U$</td>
</tr>
<tr>
<td>$P_B$</td>
<td>$-(1 - R_B)V^- - U$</td>
<td>$-(1 - R_C)V^+ - U$</td>
<td>$-(1 - R_C)V^+ - U$</td>
</tr>
<tr>
<td>Deposit @ $r$</td>
<td>$-R_B V^- $</td>
<td>$-R_B V^- $</td>
<td>$-R_B V^- $</td>
</tr>
<tr>
<td>I: borrow @ $r$</td>
<td>$-V^+$</td>
<td>$-V^+$</td>
<td>$-V^+$</td>
</tr>
<tr>
<td>II: borrow @ $r_F$</td>
<td>$-V^+$</td>
<td>$-V^+$</td>
<td>$-V^+$</td>
</tr>
<tr>
<td>Total hedge (I)</td>
<td>$-(V + U)$</td>
<td>$(1 - R_C)V^+ + U$</td>
<td>$(1 - R_C)V^+ + U$</td>
</tr>
<tr>
<td>Total hedge (II)</td>
<td>$-(V + U)$</td>
<td>$-(1 - R_B)V^- - U$</td>
<td>$-(1 - R_B)V^- - U$</td>
</tr>
<tr>
<td>Derivative</td>
<td>$V + U$</td>
<td>$V + R_C V^+$</td>
<td>$R_B V^- + V^+$</td>
</tr>
<tr>
<td>Total $\hat{V} + \Pi$ (I)</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Total $\hat{V} + \Pi$ (II)</td>
<td>$0$</td>
<td>$0$</td>
<td>$(1 - R_B)V^+$</td>
</tr>
</tbody>
</table>

- **Case I:** Derivative **can** be posted as collateral so $s_F = 0$.
- **Case II:** Derivative **cannot** be posted as collateral so $s_F > 0$. 


Credit and funding adjustments

The Ansatz $\hat{V} = V + U$ gives the following PDE for $U$:

$$\partial_t U + A_t U - (r + \lambda_B + \lambda_C) U = (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+ + s_F V^+$$

$$U(T, S) = 0$$

Solution by the Feynman-Kac theorem:

$$U(t, S) = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B + \lambda_C}(t, u) \mathbb{E}_t \left[ V^+(u, S(u)) \right] du$$

$$- (1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B + \lambda_C}(t, u) \mathbb{E}_t \left[ V^-(u, S(u)) \right] du$$

$$- \int_t^T s_F(u) D_{r+\lambda_B + \lambda_C}(t, u) \mathbb{E}_t \left[ V^+(u, S(u)) \right] du$$
Credit and funding adjustments (ctd)

Decompose $U$ further as $U = CVA + DVA + FCA$ with

- **CVA**: (modified unilateral) credit value adjustment
- **DVA**: debt value adjustment
- **FCA**: funding cost adjustment

Some comments:

- **CVA**: is modified as it is conditioned to issuer not defaulting first
- **DVA**: excess earned by issuer when buying back own bonds out of the positive cash account
- thus this is a funding benefit that monetises ”own counterparty risk” without defaulting
- **FCA**: cost if issuer has to use unsecured funding for negative cash account
Case 1: $s_F = 0$

If the derivative can be used as collateral:

- perfect hedging in all scenarios \textit{including} own default
- $s_F = 0$ and, therefore, FCA vanishes
- the adjustment $U$ is equal to the \textit{classical bilateral CVA} (see e.g. Gregory [4])
- symmetric prices
- have \textbf{justified} issuer hedging own credit \textbf{via repurchase of his own bonds}
  - where repurchase is funded through cash account
- but it’s difficult in practise to use the derivative as collateral
Case II: $s_F > 0$

If the derivative cannot be used as collateral:
- perfect hedging in all scenarios except own default
- hedge error always a windfall to the issuer
- windfall cannot be monetised by issuer...
- ...so it appears as the additional FCA cost term in the valuation

Implications
- non-Symmetric derivatives prices so $\hat{V}$ has to be re-interpreted as the issuer production cost
- the market-price will be given by supply and demand
- issuers with lower funding costs than the market clearing level will make excess profits (c.f. the electricity market)
- exact size of the FCA linked to the details of the issuer funding strategy
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Balance sheet impact and windfall

Consider simple balance sheet model without the derivative:

- expected recovery: \( R_0 = \frac{A_0}{L_0} \)
- hazard rate: \( \lambda \)
- funding spread: \( (1 - R_0)\lambda \)
- funding cost per unit time: \( f_0 = (1 - R_0)\lambda L_0 \)

Now add the derivative \( d \) to the balance sheet:

- add asset \( d \) and liability \( d \) (derivative value un-affected by the default)
- expected recovery after adding \( d \): \( R_1 = \frac{A_1}{L_1} = \frac{A_0 + d}{L_0 + d} \)
- new funding cost

\[
f_1 dt = (r + (1 - R_1)\lambda)L_1 dt
\]
\[
= r \cdot d \cdot dt + f_0 dt
\]
Implications:

- effective cost of funding for derivative is at risk free rate $r$
- this is because the counterparty has to pay back $d$ in full in cash under ISDA
- hard to make this link in practise
- more direct ways to monetize this windfall and reduce the FCA?
- using derivative as collateral is one (see previous discussion)
  - again, in practise not that straightforward
- managing balance sheet impact is another one
  - possible if one can fund issuing/repurchasing 2 bonds of different recovery
Balance sheet management of funding impact

If issuer can freely issue/repurchase own bonds $P_1$ and $P_2$ of different recoveries $0 \leq R_1 < R_2 \leq 1$:

$$
\begin{cases}
\frac{dP_1}{P_1} = r_1(t)dt - (1 - R_1)dJ_B \\
\frac{dP_2}{P_2} = r_2(t)dt - (1 - R_2)dJ_B
\end{cases}
$$

Replicating portfolio ($S$ and $P_C$ financed via repo)

$$
\Pi = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \delta S + \beta_S + \beta_C
$$

$$
-\hat{V} = \alpha_1 P_1 + \alpha_2 P_2
$$

Own default hedged with

$$
\alpha_1 = -\frac{R_2 \hat{V} - V^+ - R_B V^-}{(R_2 - R_1)P_1}
$$

$$
\alpha_2 = -\frac{V^+ + R_B V^- - R_1 \hat{V}}{(R_2 - R_1)P_2}
$$
Balance sheet management of funding impact (ctd.)

Pricing PDE if no basis (i.e. $r_i = r + (1 - R_i)\lambda_B$ for $i = 1, 2$):

$$\partial_t \hat{V} + A_t \hat{V} - r \hat{V} = - (R_B \lambda_B + \lambda_C) V^- - (\lambda_B + R_C \lambda_C) V^+$$

Implications:

- same PDE as for $s_F = 0$ in previous section (not surprising since perfect hedging)
- hence $\hat{V} = V + U$ with $U$ being the classical bilateral CVA
- issuing senior $P_2$–bonds and buying junior $P_1$–bonds not always possible.
- typically requires posting the derivative as collateral.

What happens if $P_2$ cannot be issued?
A model with one own bond only

If issuer has cannot issue $P_2$ then $\hat{V} = -\alpha_1 P_1$

- raise all funds by issuing $P_1$ by necessity
- invest all cash by purchasing $P_1$ since it offers higher return than $P_2$
- no degrees of freedom left to try to hedge own default
- hedge out the share and counterparty default with $S$ and $P_C$ as usual

Pricing PDE and decomposition $\hat{V} = V + U$

$$\partial_t \hat{V} + A_t \hat{V} - (r_1 + \lambda_C) \hat{V} = -\lambda_C (R_C V^+ + V^-)$$

$$U(t, S) = -(1 - R_C) \int_t^T \lambda_C(u) D_{r_1 + \lambda_C}(t, u) \mathbb{E}_t [V^+(u, S(u))] \, du$$

$$\quad - \int_t^T s_1(u) D_{r_1 + \lambda_C}(t, u) \mathbb{E}_t [V(u, S(u))] \, du$$
A model with one own bond only (ctd)

Some model properties:

- setting $\lambda_C = 0$ retrieves the model in Piterbarg [5] for an un-collateralised trade.
  - straightforward to add collateral to the models in this presentation
- the formula for $U$ is different from the earlier setup but has the same structure
- hedge error at own default given by $(1 - R_B)V^+ - R_B U$ (windfall or shortfall)
- the $P_1$ bond is not able to offset the disappearance of $U$ from the valuation
- no direct reference to the issuer hazard rate $\lambda_B$
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Considered hedging strategies for bilateral counterparty risk in the presence of funding costs.

Perfect hedging in all scenarios imply risk-free financing, the classical bilateral CVA and symmetric prices

- but this probably requires the issuer to be able to post the derivative as collateral.

More realistic hedging strategies are imperfect:

- not hedged in scenarios at own default
- impossibility to monetise short/windfall implies additional funding cost terms...
- ...which size depends on the precise funding/hedging strategy used.
- asymmetric prices

**Details** see Burgard and Kjaer [2], [3].


